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APPROXIMATE DISTRIBUTION FUNCTIONS
OF CERTAIN SAMPLE SERIAL CORRELATION COEFFICIENTS

A THESIS
SUBMITTED TO THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE
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by

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FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Approximate Distribution Functions of Certain Sample Serial Correlation Coefficients", submitted by Charles Algernon Patrick B.Sc. in partial fulfilment of the requirements for the degree of Master of Science.

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ABSTRACT

This study is concerned with the derivation of approximate distribution functions for sample serial correlation coefficients in linear Markov processes. Two cases are considered: first, when the defined process is circularly correlated, and second, when the process is stationary to the second order. Chapter one contains a review of the literature relevant to this special problem. In Chapter two, certain results, preliminary to the application of asymptotic procedures, are established. Asymptotic expansions for the distribution functions are derived in Appendix three, following the techniques developed by A. Erdélyi and M. Wyman, and in Chapter three these results are used to obtain asymptotic representations for the distribution functions.

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CHAPTER I

INTRODUCTION

Even a cursory survey of statistical literature suffices to indicate that the distribution problem for serial correlation coefficients is by no means of recent origin. A considerable volume of work dealing with various aspects of this problem has been published since 1940, and as a continuation of this, we, in this thesis, deal with one specialized aspect of this problem. Specifically, we concern ourselves with the derivation, by asymptotic procedures, of approximate expressions for distribution functions for sample estimates of serial correlation coefficients in linear Markov processes.

The first application of asymptotic methods to this problem was made in DANIELS (1956), where Debye's method of steepest descents was used to obtain approximations for probability density functions. All prior, and a majority of subsequent, publications deal with variations of this problem by non-asymptotic procedures. Since in the ensuing discussion we shall only be concerned with this specialized point of view, that is, the application of asymptotic procedures to the distribution problem, we forego a survey of the literature in the non-asymptotic setting, referring the interested reader to PATTON (1961) and ABBASSI (1964) for comprehensive summaries.

We now give a brief resumé of the development in DANIELS (1956).

Consider an ordered set (n -tuple) of random variables, which is assumed to be a sample, drawn at times $t_1, \dots, t_n$, from the linear Markov

process defined by

$$x_s - \alpha x_{s-1} = \epsilon_s$$

where $s = 1, \dots, n$, α is a real constant, called the lag one serial correlation coefficient with $|\alpha| < 1$. The $\epsilon_1, \dots, \epsilon_n$ are independent random variables, each having a normal distribution with mean zero and variance one. There is imposed the further restriction that the state of the process at any given time be independent of assumptions regarding an initial state. The devices used by Daniels to realize this restriction are two: first, following a suggestion by Hotelling, he assume that the process defined is circularly orientated, that is, the sample observation at time t_{n+1} is taken to be identical with the sample observation at time t_1 , for any predetermined finite integer n . This artificial restriction not only obviates any necessity for the consideration of a possible initial state, but also simplifies the subsequent analysis of the distribution problem. In the second case, Daniels assumes that the process defined is stationary to the second order [which, by virtue of the normality of the ϵ 's, implies complete stationarity]. This assumption, in contrast to the circularity restriction, is a natural one, if it is assumed that the process has been in progress for a considerable period of time.

In the circular case, the joint distribution of the x 's is multivariate normal, and is given by

$$dF(x_1, \dots, x_n) = \frac{1-\alpha^n}{(2\pi)^{\frac{n}{2}}} \exp\left[-\frac{1}{2}\{(1+\alpha^2)(x_1^2 + \dots + x_n^2) - 2\alpha(x_1x_2 + x_2x_3 + \dots + x_{n-1}x_n + x_nx_1)\}\right] dx_1 \dots dx_n,$$

and the maximum likelihood estimator

$$\alpha = \frac{x_1 x_2 + x_2 x_3 + \dots + x_{n-1} x_n + x_n x_1}{x_1^2 + \dots + x_n^2} = \frac{C_{11}}{C_{01}}$$

is used as the sample estimate of α .

In the non-circular case, that is, in the stationary case, the joint distribution of the x 's is also multivariate normal, and is given by

$$dF(x_1, \dots, x_n) = \frac{(1-\alpha^2)^{\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}} \exp\left[-\frac{1}{2}\{x_1^2 + (1+\alpha^2)(x_2^2 + \dots + x_{n-1}^2) + x_n^2 - 2\alpha(x_1 x_2 + \dots + x_{n-1} x_n)\}\right] dx_1 \dots dx_n,$$

and, the intra-class correlation coefficients between the sequences

$\{x_1, \dots, x_{n-1}\}$ and $\{x_2, \dots, x_n\}$,

$$\alpha = \frac{x_1 x_2 + \dots + x_{n-1} x_n}{\frac{1}{2}x_1^2 + x_2^2 + \dots + x_{n-1}^2 + \frac{1}{2}x_n^2} = \frac{C_{12}}{C_{02}}$$

is used as sample estimate of α .

By a consideration of the joint moment-generating function of C_0 and C_1 , together with an application of the Cramér—Geary inversion formula, Daniels derives approximations for density function of the ratios $\alpha = C_1/C_0$ in each of the above cases. Explicitly, the expressions extracted are, for the circular case

$$h(r) [= \Pr(\alpha=r)] \sim \frac{\Gamma(\frac{n}{2} + 1)}{\sqrt{\pi} (\frac{n}{2} + \frac{1}{2})} \frac{(1-r^2)^{\frac{n}{2} - \frac{1}{2}}}{(1-2\alpha r + \alpha^2)^{\frac{n}{2}}},$$

and for the stationary case,

$$h(r) [= \Pr(\alpha=r)] \sim \frac{n}{2\sqrt{\pi}} \frac{\Gamma(\frac{n}{2} - 1)}{\Gamma(\frac{n}{2} - \frac{1}{2})} \frac{(1-\alpha^2)^{\frac{1}{2}} (1-r^2)^{\frac{n}{2} - 1}}{(1-\alpha r)(1-2\alpha r + \alpha^2)^{\frac{n}{2} - 1}}$$

where, in both cases, $-1 \leq r \leq +1$.

The two estimators considered are ratios of quadratic forms in $x_1, \dots, x_n$, with the numerator indefinite and the denominator positive definite, which suggests the consideration of the more general problem of obtaining expressions (approximate or otherwise) for distributions of ratios of quadratic forms, with indefinite numerators and positive definite denominators.

In GURLAND (1948), (1953), (1955) and (1956) are found various aspects of this problem in its more general setting. The approach taken in these papers is briefly as follows: C_0 and C_1 are taken to be arbitrary quadratic forms in the random variables $x_1, \dots, x_n$. The $x_1, \dots, x_n$ have a joint multivariate normal distribution, and C_0 is restricted so that it is always positive definite. The linear function $u = C_1 - rC_0$ (linear homogeneous in C_0, C_1 with $-1 \leq r \leq 1$) is considered next, which by the manner of its definition is an indefinite quadratic form in $x_1, \dots, x_n$. Then, in contrast to DANIELS (1956), the characteristic function of u is considered, and an inversion formula for this characteristic function is derived [cf GURLAND (1948) and GILPALAEZ (1951)]. The distribution function of the ratio C_1/C_0 is then derived

by setting $u = 0$ in this inversion formula. Thus, effectively, the problem in its most fundamental form is that of the derivation of distributions of indefinite quadratic forms in $x_1, \dots, x_n$.

In GURLAND (1953) and (1955), the procedure outlined above is utilized, and, convergent Laguerrian expansions for distribution functions for quadratic forms [positive definite and indefinite], and ratios of quadratic forms [denominator always positive definite] are derived. In GURLAND (1956) the same results are modified, in that convergent expansions for quadratic forms and ratios of quadratic forms [with the same restriction as before] are derived in terms of Incomplete Gamma and Incomplete Beta functions respectively.

In this work, our approach is almost exactly that of DANIELS (1956). That is, we assume the existence of linear Markov processes under the restrictions of circularity and stationarity assumptions, and, further, use as estimators of the serial correlation coefficients, the maximum likelihood estimator and the intra-class serial correlation coefficient respectively. However, in contrast to DANIELS (1956), we derive approximations for distribution function rather than density functions.

Notwithstanding the restricted nature of our work in comparison with that of Gurland, namely, that of considering special sample serial correlation coefficients rather than more general ratios of quadratic forms, and the difference in our terminal objectives, namely, that of obtaining asymptotic expressions rather than convergent expansions, our work displays definite evidence of Gurland's influence, especially in Chapter two.

Briefly, the procedure in the ensuing chapters and appendices takes the following direction.

In Chapter two, we obtain the complex characteristic function for $u = C_1 - rC_0$ [where C_0, C_1 take the meanings assigned with respect to sample serial correlation coefficients and, as before, $-1 \leq r \leq +1$]. Following techniques similar to DANIELS (1956), we utilize the analyticity properties of the characteristic function [cf WIDDER (1941) and LUKACS (1960)] to derive complex integrals which represent the distribution function for the sample serial correlation coefficient.

In Appendix three, diverging from Daniels' approach, we use techniques developed in ERDELYI and WYMAN (1963) and (1963a) to derive asymptotic expansions for the complex integral in terms of Confluent Hypergeometric functions.

Finally, in Chapter three, using the results obtained in Appendix three, we give Asymptotic representations for distribution functions of the defined sample serial correlation coefficients.

Appendices one and two are devoted to certain auxiliary results used in the developments of Chapter two, Appendix three and Chapter three.

Appendix four contains graphical illustration of the behavior of the derived distribution functions for various values of α and various sample sizes n .

CHAPTER II

PRELIMINARY RESULTS

We wish to derive, by Asymptotic Procedures, approximations to distribution functions for sample estimates of lag one, serial correlation coefficients. The stochastic process is, specifically, the linear Markov process in discrete time defined by

$$x_t - \alpha x_{t-1} = \epsilon_t$$

for all t , where the constant α is called the serial correlation coefficient and the $\{\epsilon_t\}$ are independently and identically distributed, $N(0,1)$, random variables.

§ 1. Two Special Cases:

In this thesis we confine our attention to two special cases. In the first case we assume that the process defined is circular, that is,

$$(2.1.1) \quad x_t - \alpha x_{t-1} = \epsilon_t$$

for $t = 1, \dots, n$, where, $|\alpha| < 1$ and $x_1 \equiv x_{n+1}$. In the second case we assume the process to be stationary, that is,

$$(2.1.2) \quad x_t - \alpha x_{t-1} = \epsilon_t$$

for $t = 1, \dots, n$, where $|\alpha| < 1$ and

$$\text{cov}(x_t, x_{t-s}) = \frac{\alpha^{|s|}}{1-\alpha^2}.$$

Let $x_1, \dots, x_n$ be a sample of observations from either one of the two defined processes. They have a joint multivariate normal distribution since $\epsilon_1, \dots, \epsilon_n$ are independent, $N(0,1)$, random variables.

Explicitly, for the circular case, the distribution of $x_1, \dots, x_n$ is given by

$$(2.1.3) \quad dF(x_1, \dots, x_n) = \frac{(1-\alpha^n)}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2}\{(1+\alpha^2)(x_1^2 + \dots + x_n^2) - 2\alpha(x_1x_2 + x_2x_3 + \dots + x_{n-1}x_n + x_nx_1)\}\right] dx_1 \dots dx_n ,$$

which, using the left-hand matrix notation, may be expressed as

$$(2.1.4) \quad dF(x_1, \dots, x_n) = \frac{(1-\alpha^n)}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2} \underline{x}' \underline{\Sigma}^{-1} \underline{x}\right] dx_1 \dots dx_n ,$$

where,

$$\underline{x}' = (x_1, \dots, x_n)$$

and

$$\underline{\Sigma}^{-1} = \begin{bmatrix} 1+\alpha^2 & -\alpha & 0 & 0 & \dots & 0 & 0 & 0 & -\alpha \\ -\alpha & 1+\alpha^2 & -\alpha & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -\alpha & 1+\alpha^2 & -\alpha & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 1+\alpha^2 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1+\alpha^2 & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & -\alpha & 1+\alpha^2 & -\alpha & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -\alpha & 1+\alpha^2 & -\alpha \\ -\alpha & 0 & 0 & 0 & \dots & 0 & 0 & -\alpha & 1+\alpha^2 \end{bmatrix}$$

with

$$|\underline{\Sigma}^{-1}|^{\frac{1}{2}} = 1 - \alpha^n .$$

For the non-circular case, the distribution of $x_1, \dots, x_n$ is given by

$$(2.1.5) \quad dF(x_1, \dots, x_n)$$

$$= \frac{(1-\alpha^2)^{\frac{1}{2}}}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2}\{x_1^2 + (1+\alpha^2)(x_2^2 + \dots + x_{n-1}^2) + x_n^2 - 2\alpha(x_1 x_2 + \dots + x_{n-1} x_n)\}\right]$$

$$dx_1 \dots dx_n$$

which expressed in left-hand matrix notation is

$$(2.1.6) \quad dF(x_1, \dots, x_n) = \frac{(1-\alpha^2)^{\frac{1}{2}}}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2} \underline{x}' \underline{\Sigma}^{-1} \underline{x}\right] dx_1 \dots dx_n$$

where,

$$\underline{x}' = (x_1, \dots, x_n)$$

and

$$\underline{\Sigma}^{-1} = \begin{bmatrix} 1 & -\alpha & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -\alpha & 1+\alpha^2 & -\alpha & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -\alpha & 1+\alpha^2 & -\alpha & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 1+\alpha^2 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1+\alpha^2 & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & -\alpha & 1+\alpha^2 & -\alpha & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -\alpha & 1+\alpha^2 & -\alpha \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -\alpha & 1 \end{bmatrix}$$

with

$$|\underline{\Sigma}^{-1}|^{\frac{1}{2}} = (1-\alpha^2)^{\frac{1}{2}}.$$

As an estimate of α we use, in the circular case, the function

$$(2.1.7) \quad \hat{\alpha}(x_1, \dots, x_n) = \frac{x_1^2 x_2^2 + x_2^2 x_3^2 + \dots + x_{n-1}^2 x_n^2 + x_n^2 x_1^2}{x_1^2 + x_2^2 + \dots + x_{n-1}^2 + x_n^2}$$

which in matrix notation is

$$(2.1.8) \quad \hat{\alpha} = \frac{\underline{x}' \underline{N} \underline{x}}{\underline{x}' \underline{D} \underline{x}}$$

where,

$$\underline{N} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \dots & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \dots & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

and

$$\underline{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix}$$

and in the non-circular case, the function

$$(2.1.9) \quad \hat{\alpha}(x_1, \dots, x_n) = \frac{x_1 x_2 + \dots + x_{n-1} x_n}{\frac{1}{2}x_1^2 + x_2^2 + \dots + x_{n-1}^2 + \frac{1}{2}x_n^2}$$

which in matrix notation is

$$(2.1.10) \quad \hat{\alpha} = \frac{\underline{x}' \underline{N} \underline{x}}{\underline{x}' \underline{D} \underline{x}}$$

where,

$$\underline{N} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \dots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

and,

$$\underline{D} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

In both cases

$$|\hat{\alpha}| \leq 1 .$$

Thus, the problem of finding the sampling distribution of α is effectively that of finding the distribution function for the ratio of two quadratic forms.

§ 2. The Distribution of the Ratio of two Quadratic forms.

We have the ratio

$$(2.2.1) \quad \hat{\alpha} = \frac{\underline{x}' \underline{N} \underline{x}}{\underline{x}' \underline{D} \underline{x}}$$

where,

$$\underline{x}' = (x_1, \dots, x_n) ,$$

$\underline{N}$ is an indefinite symmetric matrix of constant coefficients,

$\underline{D}$ is a positive definite symmetric matrix of constant coefficients.

Denoting the distribution function of α by $H(\cdot)$, by definition

$$(2.2.2) \quad H(r) = P_r[\alpha \leq r]$$

$$= P_r \left[\frac{\underline{x}' \underline{N} \underline{x}}{\underline{x}' \underline{D} \underline{x}} \leq r \right]$$

where, r is any real number satisfying $|r| \leq 1$. For reasons not immediately obvious, which become apparent as the discussion unfolds, we restrict r so that $|r| < 1$. Thus,

$$P_r \left[\frac{\underline{x}' \underline{N} \underline{x}}{\underline{x}' \underline{D} \underline{x}} \leq r \right] = P_r [(\underline{x}' \underline{N} \underline{x}) \leq r(\underline{x}' \underline{D} \underline{x})]$$

$$= P_r [\underline{x}' (\underline{N} - r \underline{D}) \underline{x} \leq 0] .$$

Putting

$$\underline{x}' (\underline{N} - r \underline{D}) \underline{x} = u(x_1, \dots, x_n)$$

and denoting the distribution function of u by $G(\cdot)$, we get

$$(2.2.3) \quad H(r) = P_r[\alpha \leq r] = P_r[u \leq 0] = G(0) .$$

$(\underline{N} - r \underline{D})$ is a symmetric matrix which is negative definite, singular or positive definite according as the value of r varies over the open interval $(-1, 1)$, that is, $(\underline{N} - r \underline{D})$ is an indefinite matrix of variable coefficients.

The characteristic function of $u(x_1, \dots, x_n)$ is

$$(2.2.4) \quad \varphi_u(\xi) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp[i\xi u(x_1, \dots, x_n)] dF(x_1, \dots, x_n)$$

where, ζ is a complex variable

$$u(x_1, \dots, x_n) = \underline{x}'(\underline{N} - r\underline{D}) \underline{x}$$

and

$$dF(x_1, \dots, x_n) = \frac{|\underline{Z}^{-1}|^{\frac{1}{2}}}{(2\pi)^{n/2}} \exp[-\frac{1}{2} \underline{x}' \underline{Z}^{-1} \underline{x}] dx_1 \dots dx_n .$$

$\varphi_u(\zeta)$ exists for all ζ such that $\text{Im } \zeta = 0$, but may also exist for some ζ such that $\text{Im } \zeta \neq 0$. In the former case $\varphi_u(\zeta)$ is merely a complex valued function of a real variable ($\text{Re } \zeta$) defined on the line $\text{Im } \zeta = 0$, but in the latter case $\varphi_u(\zeta)$ is an analytic function which exists and is regular in some strip of non-zero width, which contains $\text{Im } \zeta = 0$. However, since r in $\underline{x}'(\underline{N} - r\underline{D}) \underline{x}$ is not "integrated out", r behaves as a "free variable", and a continuous variation of r in $(-1, 1)$ causes the width of the strip to fluctuate. To determine the distribution function of u from the characteristic function, we use an inversion formula which is independent of this fluctuation. [Since we do not rule out the possibility of a degenerate strip for some values of r]. The inversion formula is

$$(2.2.5) \quad G(v) = \frac{1}{2} - \frac{1}{2\pi i} \oint \frac{\exp[-i\zeta v] \varphi_u(\zeta)}{\zeta} d\zeta ,$$

which, for $v = 0$, will give the distribution function of α , that is

$$(2.2.6) \quad H(r) = \frac{1}{2} - \frac{1}{2\pi i} \oint \frac{\varphi_u(\zeta)}{\zeta} d\zeta ,$$

where,

$$(2.2.7) \quad \oint = \lim_{\substack{T \rightarrow \infty \\ \epsilon \rightarrow 0}} \left(\int_{-T}^{-\epsilon} + \int_{\epsilon}^T \right)$$

the path of integration being the line $\text{Im } \zeta = 0$.

In matrix notation the characteristic function is

$$(2.2.8) \quad \begin{aligned} \varphi_u(\zeta) &= \frac{|\underline{Z}|^{-\frac{1}{2}}}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp[i\zeta \underline{x}'(\underline{N}-r\underline{D})\underline{x}] \exp[-\frac{1}{2}\underline{x}'\underline{Z}^{-1}\underline{x}] d\underline{x}_1 \dots d\underline{x}_n \\ &= \frac{|\underline{Z}|^{-\frac{1}{2}}}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp[-\frac{1}{2}\underline{x}'\{\underline{Z}^{-1}-2i\zeta(\underline{N}-r\underline{D})\}\underline{x}] d\underline{x}_1 \dots d\underline{x}_n . \end{aligned}$$

There is an orthogonal transformation that will diagonalize $\{\underline{Z}^{-1}-2i\zeta(\underline{N}-r\underline{D})\}$.

Making this transformation and integrating we get

$$(2.2.9) \quad \varphi_u(\zeta) = \frac{|\underline{Z}|^{-\frac{1}{2}}}{|\underline{Z}^{-1}-2i\zeta(\underline{N}-r\underline{D})|^{\frac{1}{2}}}$$

where

$$|\underline{Z}^{-1} - 2i\zeta(\underline{N}-r\underline{D})|^{-\frac{1}{2}}$$

is the Jacobian of the transformation.

§ 3. Characteristic Functions for Special Cases.

In this section we give explicit expressions for the characteristic functions in the two special cases considered in this thesis.

The circular case:

We have, by (2.1.4),

$$(2.3.1) \quad \underline{Z}^{-1} = \begin{bmatrix} 1+\alpha^2 & -\alpha & 0 & \dots & 0 & 0 & -\alpha \\ -\alpha & 1+\alpha^2 & -\alpha & \dots & 0 & 0 & 0 \\ 0 & -\alpha & 1+\alpha^2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1+\alpha^2 & -\alpha & 0 \\ 0 & 0 & 0 & \dots & -\alpha & 1+\alpha^2 & -\alpha \\ -\alpha & 0 & 0 & \dots & 0 & -\alpha & 1+\alpha^2 \end{bmatrix}$$

with

$$|\underline{Z}^{-1}|^{\frac{1}{2}} = 1 - \alpha^n$$

and because of (2.1.8)

$$(2.3.2) \quad \underline{N} - r\underline{D} = \begin{bmatrix} -r & \frac{1}{2} & 0 & \dots & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & -r & \frac{1}{2} & \dots & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -r & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -r & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \dots & \frac{1}{2} & -r & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \dots & 0 & \frac{1}{2} & -r \end{bmatrix}$$

Thus

$$(2.3.3) \quad \underline{Z}^{-1} = 2i\zeta(\underline{N} - r\underline{D})$$

$$= \begin{bmatrix} 1+\alpha^2+2ir\zeta & -(\alpha+i\zeta) & 0 & \dots & 0 & 0 & -(\alpha+i\zeta) \\ -(\alpha+i\zeta) & 1+\alpha^2+2ir\zeta & -(\alpha+i\zeta) & \dots & 0 & 0 & 0 \\ 0 & -(\alpha+i\zeta) & 1+\alpha^2+2ir\zeta & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1+\alpha^2+2ir\zeta & -(\alpha+i\zeta) & 0 \\ 0 & 0 & 0 & \dots & -(\alpha+i\zeta) & 1+\alpha^2+2ir\zeta & -(\alpha+i\zeta) \\ -(\alpha+i\zeta) & 0 & 0 & \dots & 0 & -(\alpha+i\zeta) & 1+\alpha^2+2ir\zeta \end{bmatrix}$$

$$= (\alpha+i\zeta)^n \begin{bmatrix} \frac{1+\alpha^2+2ir\zeta}{\alpha+i\zeta} & -1 & 0 & \dots & 0 & 0 & -1 \\ -1 & \frac{1+\alpha^2+2ir\zeta}{\alpha+i\zeta} & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & \frac{1+\alpha^2+2ir\zeta}{\alpha+i\zeta} & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \frac{1+\alpha^2+2ir\zeta}{\alpha+i\zeta} & -1 & 0 \\ 0 & \dots & -1 & \frac{1+\alpha^2+2ir\zeta}{\alpha+i\zeta} & -1 \\ -1 & \dots & 0 & -1 & \frac{1+\alpha^2+2ir\zeta}{\alpha+i\zeta} \end{bmatrix}$$

$$= (\alpha + i\zeta)^n |\underline{B}| .$$

Thus, by equation (2.2.9),

$$(2.3.4) \quad \varphi_u(\zeta) = \frac{1-\alpha^n}{(\alpha+i\zeta)^{n/2}} \frac{1}{|\underline{B}|^{\frac{1}{2}}} .$$

Now $|\underline{B}|$ is a circulant and has value

$$(2.3.5) \quad |\underline{B}| = \frac{(1-z^n)^2}{z^n}$$

where,

$$(2.3.6) \quad z + \frac{1}{z} = \frac{1+\alpha^2+2ir\zeta}{\alpha+i\zeta} .$$

Therefore,

$$(2.3.7) \quad \alpha + i\zeta = \frac{z(1-2\alpha r+\alpha^2)}{(1-2rz+z^2)}$$

and

$$(2.3.8) \quad \varphi_u(\zeta(z)) = \frac{1-\alpha^n}{(1-2\alpha r+\alpha^2)^{n/2}} \frac{(1-2rz+z^2)^{n/2}}{(1-z^n)}$$

where,

$$(2.3.8a) \quad \zeta(z) = i\alpha - \frac{iz(1-2\alpha r+\alpha^2)}{1-2rz+z^2} .$$

The non-circular case:

We have by (2.1.6)

$$(2.3.9) \quad \underline{Z}^{-1} = \begin{bmatrix} 1 & -\alpha & 0 & \dots & 0 & 0 & 0 \\ -\alpha & 1+\alpha^2 & -\alpha & \dots & 0 & 0 & 0 \\ 0 & -\alpha & 1+\alpha^2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1+\alpha^2 & -\alpha & 0 \\ 0 & 0 & 0 & \dots & -\alpha & 1+\alpha^2 & -\alpha \\ 0 & 0 & 0 & \dots & 0 & -\alpha & 1 \end{bmatrix}$$

with

$$|\underline{Z}^{-1}| = 1 - \alpha^2$$

and because of (2.1.10)

$$(2.3.10) \quad \underline{N-rD} = \begin{bmatrix} -\frac{1}{2}r & \frac{1}{2} & 0 & \dots & 0 & 0 & 0 \\ \frac{1}{2} & -r & \frac{1}{2} & \dots & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -r & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -r & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \dots & \frac{1}{2} & -r & \frac{1}{2} \\ 0 & 0 & 0 & \dots & 0 & \frac{1}{2} & -\frac{1}{2}r \end{bmatrix}.$$

Thus,

$$(2.3.11) \quad \underline{Z}^{-1} = 2i\zeta(\underline{N-rD})$$

$$= \begin{bmatrix} 1+i\zeta & -(\alpha+i\zeta) & 0 & \dots & 0 & 0 & 0 \\ -(\alpha+i\zeta) & 1+\alpha^2+2ir\zeta & -(\alpha+i\zeta) & \dots & 0 & 0 & 0 \\ 0 & -(\alpha+i\zeta) & 1+\alpha^2+2ir\zeta & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1+\alpha^2+2ir\zeta & -(\alpha+i\zeta) & 0 \\ 0 & 0 & 0 & \dots & -(\alpha+i\zeta) & 1+\alpha^2+2ir\zeta & -(\alpha+i\zeta) \\ 0 & 0 & 0 & \dots & 0 & -(\alpha+i\zeta) & 1+i\zeta \end{bmatrix}$$

$$= (\alpha+i\zeta)^n \begin{bmatrix} \frac{1+i\zeta}{\alpha+i\zeta} & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & \frac{1+\alpha^2+2ir\zeta}{\alpha+i\zeta} & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & \frac{1+\alpha^2+2ir\zeta}{\alpha+i\zeta} & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{1+\alpha^2+2ir\zeta}{\alpha+i\zeta} & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & \frac{1+\alpha^2+2ir\zeta}{\alpha+i\zeta} & -1 \\ 0 & 0 & 0 & \dots & 0 & -1 & \frac{1+i\zeta}{\alpha+i\zeta} \end{bmatrix}$$

$$= (\alpha+i\zeta)^n |\underline{B}^*| .$$

Thus, by equation (2.2.9)

$$(2.3.12) \quad \varphi_u(\zeta) = \frac{(1-\alpha^2)}{(\alpha+i\zeta)^{n/2}} \cdot \frac{1}{|\underline{B}^*|^{\frac{1}{2}}}.$$

Now $|\underline{B}^*|$ is a continuant, and has value

$$(2.3.13) \quad |\underline{B}^*| = \frac{\{(1-\alpha z)^2(1-\alpha r+\alpha z-rz)^2 - z^{2n-2}(\alpha-z)^2(\alpha-r+z-\alpha rz)^2\}}{z^n(1-z^2)(1-2\alpha r+\alpha^2)^2}$$

where,

$$(2.3.14) \quad z + \frac{1}{z} = \frac{1+\alpha^2+2ir\zeta}{\alpha+i\zeta}.$$

Therefore,

$$(2.3.15) \quad (\alpha+i\zeta) = \frac{z(1-2\alpha r+\alpha^2)}{1-2rz+z^2}$$

and

$$(2.3.16) \quad \varphi_u(\zeta(z)) = \frac{(1-\alpha^2)^{\frac{1}{2}}}{(1-2\alpha r+\alpha^2)^{\frac{n}{2}-1}} \times \frac{(1-z^2)^{\frac{1}{2}}(1-2rz+z^2)^{n/2}}{\{(1-\alpha z)^2(1-\alpha r+\alpha z-rz)^2 - z^{2n-2}(\alpha-z)^2(\alpha-r+z-\alpha rz)^2\}^{\frac{1}{2}}}$$

where

$$(2.3.16a) \quad \zeta(z) = i\alpha - \frac{iz(1-2\alpha r+\alpha^2)}{(1-2rz+z^2)}.$$

Now,

$$(2.3.17) \quad \frac{\xi'(z)}{\xi(z)} = \frac{(1-2\alpha r + \alpha^2)(1-z^2)}{(1-2rz + z^2)(1-\alpha z)(z-\alpha)}.$$

Thus, corresponding to (2.3.7) we have

$$(2.3.18) \quad \frac{\varphi_u(\xi(z))\xi'(z)}{\xi(z)} = \frac{1-\alpha^n}{(1-2\alpha r + \alpha^2)^{\frac{n}{2}-1}} \frac{(1-z^2)(1-2rz + z^2)^{\frac{n}{2}-1}}{(z-\alpha)(1-\alpha z)(1-z^n)}$$

and, corresponding to (2.3.16) we have

$$(2.3.19) \quad \frac{\varphi_u(\xi(z))\xi'(z)}{\xi(z)} = \frac{(1-\alpha^2)^{\frac{1}{2}}}{(1-2\alpha r + \alpha^2)^{\frac{n}{2}-1}} \frac{(1-z^2)^{\frac{3}{2}}}{(z-\alpha)(1-\alpha z)} \\ \times \frac{(1-2rz + z^2)^{\frac{n}{2}-1}}{\{(1-\alpha z)^2(1-\alpha r + \alpha z - rz)^2 + z^{2n-2}(\alpha-z)^2(\alpha-r+z-\alpha rz)^2\}^{\frac{1}{2}}}.$$

Since we are prepared to tolerate errors of $O(\lambda^n)$ [$|\lambda| < 1$] in the final approximation, for $(1-r^2)$ not small, we omit

$$\frac{1-\alpha^n}{1-z^n} \quad \text{in (2.3.18)}$$

and replace

$$\{(1-\alpha z)^2(1-\alpha r + \alpha z - rz)^2 + z^{2n-2}(\alpha-z)^2(\alpha-r+z-\alpha rz)^2\}^{\frac{1}{2}}$$

by,

$$(1-\alpha z)(1-\alpha r + \alpha z - rz) \quad \text{in (2.3.19)}.$$

Thus, neglecting terms which are exponentially small, (2.3.18) and (2.3.19) become respectively

$$(2.3.20) \quad \frac{\varphi_u(\zeta(z))\zeta'(z)}{\zeta(z)} \sim \frac{1}{(1-2\alpha r+\alpha^2)^{\frac{n}{2}-1}} \frac{(1-z^2)(1-2rz+z^2)^{\frac{n}{2}-1}}{(z-\alpha)(1-\alpha z)}$$

and

$$(2.3.21) \quad \frac{\varphi_u(\zeta(z))\zeta'(z)}{\zeta(z)} \sim \frac{(1-\alpha^2)^{\frac{1}{2}}}{(1-2\alpha r+\alpha^2)^{\frac{n}{2}-2}} \frac{(1-z^2)^{\frac{3}{2}}(1-2rz+z^2)^{\frac{n}{2}-1}}{(z-\alpha)(1-\alpha z)^2(1-\alpha r+\alpha z-rz)} .$$

§ 4. Discussion of a Conformal Transformation.

In the last section we saw that the mapping

$$(2.4.1) \quad \frac{1+\alpha^2+2ir\zeta}{\alpha+i\zeta} = z + \frac{1}{z}$$

transformed

$$(2.4.2) \quad \varphi_u(\zeta) d\zeta = \frac{|\underline{z}|^{-\frac{1}{2}}}{|\underline{z}^{-1} - 2ir\zeta(\underline{N}-r\underline{D})|^{\frac{1}{2}}} d\zeta$$

into an expression of the form

$$(2.4.3) \quad \frac{\varphi_u(\zeta(z))\zeta'(z)}{\zeta(z)} dz \sim \frac{K'}{(z-\alpha)} g(z) (1-2rz+z^2)^s dz$$

where,

- 1) $\xi = \xi(z)$ represents the transformation (2.4.1),
- 2) K' is a function of (α, r) ,
- 3) $g(z)$ is an algebraic function taking real values for z real and is regular in a simply-connected subset of $|z| \leq 1$, which contains $z = \alpha$ and $z = r$,
- 4) s is a real variable taking large values $[s = \frac{n}{2} - 1]$.

[Corresponding to (2.3.20)

$$K' = (1 - 2\alpha r + \alpha^2)^{-\frac{n}{2} + 1}$$

and

$$g(z) = (1 - z^2)/(1 - \alpha z)$$

and to (2.3.21)

$$K' = \frac{(1 - \alpha^2)^{\frac{1}{2}}}{(1 - 2\alpha r + \alpha^2)^{\frac{n}{2} - 2}}$$

and

$$g(z) = \frac{(1 - z^2)^{\frac{3}{2}}}{(1 - \alpha z)^2 (1 - \alpha r + \alpha z - rz)} \quad]$$

Equation (2.4.1) maps the ξ -plane cut from $-i\infty$ to $\frac{-1(1-\alpha)^2}{2(1-r)}$ and

from $\frac{i(1+\alpha)^2}{2(1+r)}$ to $+\infty$ onto the open disc $|z| < 1$, with the cut being mapped onto the unit-circle $|z| = 1$. The point $\zeta = \frac{-i(1-\alpha)^2}{2(1-r)}$ corresponds to the point $z = 1$ and the point $\zeta = \frac{i(1+\alpha)^2}{2(1+r)}$ corresponds to the point $z = -1$. Further, $\zeta = -\infty + i\tau$ and $\zeta = +\infty + i\tau'$ (τ, τ' arbitrary) are mapped onto $z = r - i\sqrt{1-r^2}$ and $z = r + i\sqrt{1-r^2}$ respectively.

The z -image of $\text{Im } \zeta = 0$ is the curve given by the equation

$$(2.4.4) \quad 2\alpha(z\bar{z})^2 - (1+2\alpha r + \alpha^2)z\bar{z}(z+\bar{z}) + 4r(1+\alpha^2)z\bar{z} + 2\alpha(z^2 + \bar{z}^2) - (1+2\alpha r + \alpha^2)(z+\bar{z}) + 2\alpha = 0,$$

which, in cartesian coordinates with $z = x + iy$, is

$$(2.4.5) \quad \alpha(x^2 + y^2)^2 - (1+2\alpha r + \alpha^2)x(x^2 + y^2) + 2r(1+\alpha^2)(x^2 + y^2) + 2\alpha(x^2 - y^2) - (1+2\alpha r + \alpha^2)x + \alpha = 0$$

This curve, which we shall henceforth denote by Γ , is symmetric about $\text{Im } z = 0$, which is normal to the curve at the point of intersection $z = \alpha$ [the z -image of the point $\zeta = 0$]. Thus, points below $\text{Im } z = 0$ lying on Γ are complex conjugates of points lying on the curve above the line $\text{Im } z = 0$. Γ runs from the point $z = r - i\sqrt{1-r^2}$ [the z -image of the point $\text{Re } \zeta = -\infty$] to the point $z = r + i\sqrt{1-r^2}$ [the z -image of the point $\text{Re } \zeta = +\infty$].

Thus we define the z -correspondent of the ζ -integral in the inversion formula (2.4.6) to be

$$(2.4.6) \quad \frac{K'}{2\pi i} \int_{[\Gamma]} \frac{1}{(z-\alpha)} g(z) (1-2rz+z^2)^s dz \quad \left[= \frac{1}{2\pi i} \oint \frac{\varphi_u(\zeta)}{\zeta} d\zeta \right]$$

where,

$$(2.4.7) \quad f_{[\Gamma]} = \lim_{\substack{\Gamma \\ a \rightarrow \alpha \\ z^* \rightarrow r + i\sqrt{1-r^2}}} \left[\int_{\bar{z}^*}^{\bar{a}} + \int_a^{z^*} \right] .$$

The integrals defined are curvilinear on Γ , a and its complex conjugate $\bar{a}$ are points on Γ as are z^* and $\bar{z}^*$, and the notation

$\xrightarrow{E}$

here, and in the subsequent discussion denotes passage to the limit through the set E . Thus the z -integral exists in the sense of a Cauchy Principle value as defined in Appendix I.

Now by virtue of (2.3.7) and (2.3.16), it is clear that

$$|g(z)| < \infty \quad \text{for} \quad z = r \pm i\sqrt{1-r^2} .$$

Thus

$$(2.4.8) \quad \lim_{z \rightarrow r + i\sqrt{1-r^2}} \frac{1}{z-\alpha} g(z) (1-2rz+z^2)^s = 0 ,$$

and

$$(2.4.9) \quad \lim_{z \rightarrow r - i\sqrt{1-r^2}} \frac{1}{z-\alpha} g(z) (1-2rz+z^2)^s = 0 .$$

Thus we write

$$(2.4.10) \quad f_{[\Gamma]} = \lim_{a \rightarrow \alpha} \left[\int_{\bar{A}}^{\bar{a}} + \int_a^A \right]$$

where $A = r + i\sqrt{1-r^2}$.

§ 5. Deformation of the Path of Integration.

We are concerned with integrals of the form

$$(2.5.1) \quad \frac{K'}{2\pi i} \oint_{[\Gamma]} \frac{1}{(z-\alpha)} g(z) (1-2rz+z^2)^s dz ,$$

where,

$$\oint_{[\Gamma]} = \lim_{a \rightarrow \alpha} \left[\int_{\bar{A}}^{\bar{a}} + \int_a^A \right] , \quad (A = r+i\sqrt{1-r^2})$$

and

$g(z)$ is regular in $|z| < 1$.

The integrand in (2.5.1) has critical point at $z = r$, so that we deform the path of integration to pass through $z = r$. Three special cases have to be considered. They are, $r < \alpha$, $r = \alpha$, $r > \alpha$. For the present, we shall deal with $r < \alpha$ and $r > \alpha$ only, leaving the consideration of $r = \alpha$ to be dealt with at a later stage.

The deformation of the path is always made in a manner such that

$$(2.5.2) \quad |1 - 2rz + z^2| \leq (1 - r^2)$$

for all z on the deformed path, which we denote by L . Since the integrand is regular for all z , except $z = \alpha$, in $|z| < 1$, no residue terms will

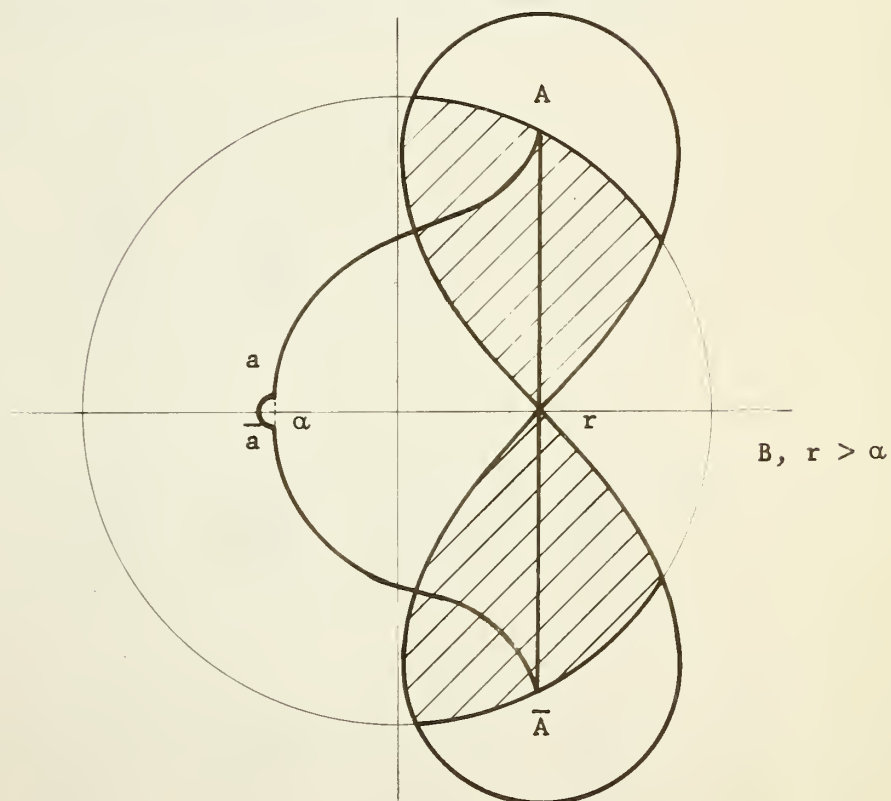
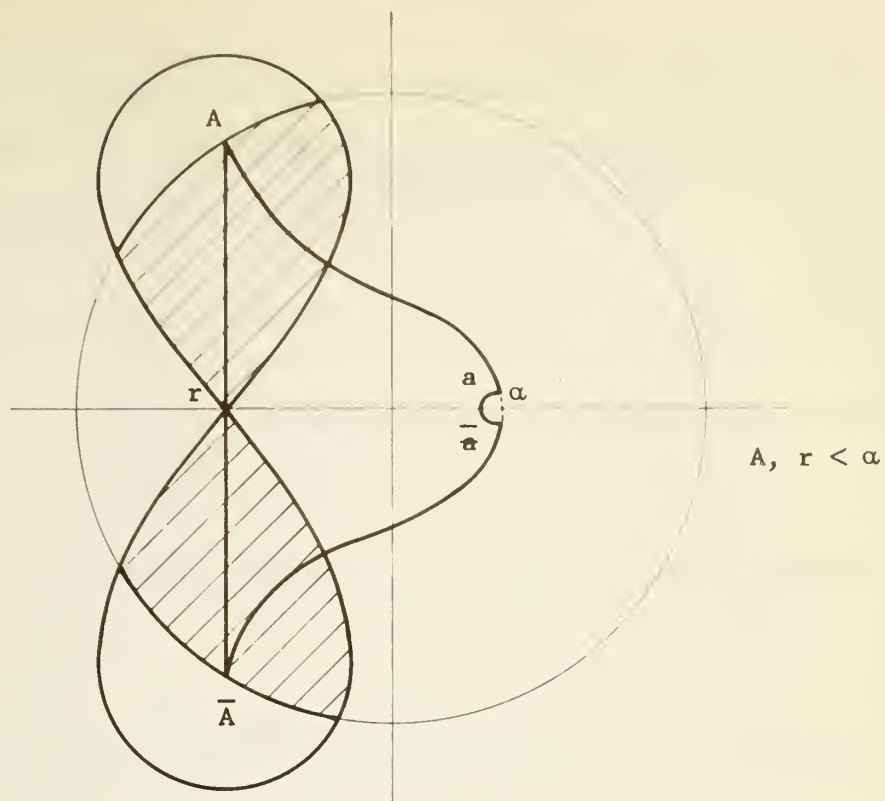


Fig 2.1
Admissible Deformed Paths of Integration

occur, except one corresponding to $z = \alpha$ [since this is a simple pole of the integrand]. Now referring to Figures 2.1(A) and 2.1(B) we make the following assumptions.

- (i) Γ represents the curve $\bar{A} \bar{a} \alpha a A$. (as before)
- (ii) L represents the curve $\bar{A} r A$ [in particular, we choose the straight line from $\bar{A}$ to A .]
- (iii) C represents the curve $\bar{a} a$ [the indentation around $z = \alpha$].
- (iv) $\int_{x_1}^{x_2}$, as before, represents the integral from x_1 to x_2 on the appropriate curve signified in [].

Thus, for $r < \alpha$

$$\int_{\bar{A}}^{\bar{a}} [\Gamma] + \int_{\bar{a}}^a [C] + \int_a^A [\Gamma] + \int_A^{\bar{A}} [L] = 0, \text{ so that}$$

$$\int_{\bar{A}}^{\bar{a}} [\Gamma] + \int_a^A [\Gamma] = \int_{\bar{A}}^A [L] + \int_a^{\bar{a}} [C].$$

Thus,

$$\lim_{a \rightarrow \alpha} \left[\int_{\bar{A}}^{\bar{a}} [\Gamma] + \int_a^A [\Gamma] \right] = \int_{\bar{A}}^A [L] + \lim_{a \rightarrow \alpha} \int_a^{\bar{a}} [C].$$

Now,

$$\lim_{\substack{C \\ a \rightarrow \alpha}} \int_a^{\bar{a}} \frac{1}{(z-\alpha)} g(z) (1-2rz+z^2)^s \\ = \pi i g(\alpha) (1-2\alpha r + \alpha^2)^s ,$$

and therefore,

$$(2.5.3) \quad \frac{K'}{2\pi i} \oint_{[\Gamma]} \frac{1}{z-\alpha} g(z) (1-2rz+z^2)^s dz \\ = \frac{K'}{2} g(\alpha) (1-2\alpha r + \alpha^2)^s + \frac{K'}{2\pi i} \int_{\substack{\bar{A} \\ [L]}}^A \frac{1}{(z-\alpha)} g(z) (1-2rz+z^2)^s dz .$$

For $r > \alpha$

$$\int_{\substack{\bar{A} \\ [\Gamma]}}^{\bar{a}} + \int_{\substack{\bar{a} \\ [C]}}^a + \int_{\substack{a \\ [\Gamma]}}^A + \int_{\substack{A \\ [L]}}^{\bar{A}} = -2\pi i (g(\alpha) (1-2\alpha r + \alpha^2)^s) ,$$

so that

$$\int_{\substack{\bar{A} \\ [\Gamma]}}^{\bar{a}} + \int_{\substack{a \\ [\Gamma]}}^A = \int_{\substack{\bar{A} \\ [L]}}^A + \int_{\substack{a \\ [C]}}^{\bar{A}} = 2\pi i (g(\alpha) (1-2\alpha r + \alpha^2)^s) .$$

Thus

$$(2.5.4) \quad \frac{K'}{2\pi i} \oint_{[\Gamma]} \frac{1}{z-\alpha} g(z) (1-2rz+z^2)^s dz \\ = - \frac{K'}{2} g(\alpha) (1-2\alpha r + \alpha^2)^s + \frac{K'}{2\pi i} \int_{\substack{\bar{A} \\ [L]}}^A \frac{1}{(z-\alpha)} g(z) (1-2rz+z^2)^s dz .$$

Now, by Appendix II

$$K' g(\alpha)(1-2\alpha r+\alpha^2)^s = 1$$

so that (2.5.3) for $r < \alpha$, becomes

$$\begin{aligned} (2.5.5) \quad \frac{K'}{2\pi i} \oint_{[\Gamma]} \frac{1}{(z-\alpha)} g(z)(1-2rz+z^2)^s dz \\ = \frac{1}{2} + \frac{K'}{2\pi i} \int_{\bar{A}}^A \frac{1}{(z-\alpha)} g(z)(1-2rz+z^2)^s dz \\ [L] \end{aligned}$$

and, (2.5.4) for $r > \alpha$, becomes

$$\begin{aligned} (2.5.6) \quad \frac{K'}{2\pi i} \oint_{[\Gamma]} \frac{1}{(z-\alpha)} g(z)(1-2rz+z^2)^s dz \\ = -\frac{1}{2} + \frac{1}{2\pi i} \int_{\bar{A}}^A \frac{1}{(z-\alpha)} g(z)(1-2rz+z^2)^s dz , \\ [L] \end{aligned}$$

and the inversion formula (2.2.6) (using (2.4.6)) for $r < \alpha$ becomes

$$(2.5.7) \quad H(r) = \frac{K'}{2\pi i} \int_{\bar{A}}^A \frac{1}{(\alpha-z)} g(z)(1-2rz+z^2)^s dz , \\ [L]$$

and, for $r > \alpha$ becomes,

$$(2.5.8) \quad H(r) = 1 - \frac{K'}{2\pi i} \int_{\bar{A}}^A \frac{1}{(z-\alpha)} g(z)(1-2rz+z^2)^s dz , \\ [L]$$

where L is the straight line from $\bar{A}$ to A .

§ 6. Change in the Variable of Integration.

The integral

$$(2.6.1) \quad \frac{K'}{2\pi i} \int_{\bar{A}}^A \frac{1}{(\alpha-z)} g(z)(1-2rz+z^2)^s dz$$

[L]

where, L is the line from $z = \bar{A} = r - i\sqrt{1-r^2}$ to $z = A = r + i\sqrt{1-r^2}$ passing through $z = r$, has a critical point at $z = r$. We change the variable of integration so that the critical point is at the origin in terms of the new variable of integration. We use the linear transformation

$$(2.6.2) \quad z = r + i\sqrt{1-r^2} t$$

so that,

$$(2.6.3) \quad dz = i\sqrt{1-r^2} dt ,$$

$$(2.6.4) \quad (\alpha-z) = \alpha - r - i\sqrt{1-r^2} t ,$$

$$(2.6.5) \quad (1-2rz+z^2) = (1-r^2)(1-t^2) ,$$

$$(2.6.6) \quad g(z) = g(r + i\sqrt{1-r^2} t) = \chi(t) ,$$

and L^* , the t -image of L , is the straight line from $t = -1$ to $t = +1$ passing through $t = 0$ [i.e., a segment of $\text{Im } t = 0$].

Thus, the t -integrals corresponding to (2.5.7) and (2.5.8) are for $r < \alpha$

$$(2.6.7) \quad H(r) = \frac{K'(1-r^2)^{s+\frac{1}{2}}}{2\pi} \int_{-1}^{+1} \frac{1}{(\alpha-r-i\sqrt{1-r^2}t)} \chi(t) (1-t^2)^s dt$$

[L*]

and, for $r > \alpha$

$$(2.6.8) \quad H(r) = 1 - \frac{K'(1-r^2)^{s+\frac{1}{2}}}{2\pi} \int_{-1}^{+1} \frac{1}{(r-\alpha+i\sqrt{1-r^2}t)} \chi(t) (1-t^2)^s dt .$$

[L*]

CHAPTER III

THE ASYMPTOTIC APPROXIMATION FOR DISTRIBUTION FUNCTIONS OF TWO SAMPLE SERIAL CORRELATION COEFFICIENTS

In this chapter, we present the asymptotic representations of certain integrals, which represent distribution functions of sample serial correlation coefficients. The method used in obtaining the necessary asymptotic expansions is the same as that which is presented in a joint paper by A. ERDELYI and M. WYMAN (1963) entitled "The Method of Laplace." Notwithstanding the essential similarity in the developments, there is one significant modification made in our work. Whereas, in Erdelyi and Wyman the basic functions involved in the Asymptotic expansions is the Euler integral for the Gamma function, we use, as basic function for the expansion, the Euler integral for the Confluent Hypergeometric function.

To preserve continuity in the ensuing presentation, the derivation of the complete expansion and other details are relegated to Appendix III. We shall draw upon the major results obtained therein, always with appropriate reference.

§ 1. Basic Assumptions and an Asymptotic Representation.

The integral for which we seek an asymptotic representation is

$$(3.1.1) \quad \int_{-1}^{+1} \frac{\chi(t)}{\alpha - r - i\sqrt{1-r^2}t} (1-t^2)^s dt$$

[L*]

where,

- (i) L^* is the line $\text{Im } t = 0$,
- (ii) s is a real variable assuming large values,
- (iii) α, r are real variables, α fixed and such that $|\alpha| < 1$,
and r varying continuously over the set $\{\xi: |\xi| < 1 \text{ and } \xi \neq \alpha\}$,
- (iv) $\chi(t)$ is an analytic function of t regular in some
neighbourhood of $t = 0$ and $|\chi(t)| < \infty$ for all t in
 $[-1, +1]$ i.e. on the path of integration.

By Appendix III,

$$\begin{aligned}
 (3.1.2) \quad & \int_{\substack{-1 \\ [L^*]}}^{+1} \frac{\chi(t)}{\alpha - r - i \sqrt{1-r^2} t} (1-t^2)^s dt \\
 &= \frac{\alpha-r}{1-r^2} \sum_{m=0}^M \sum_{k=0}^m c_k^{(m,1)} \Gamma(m+k+\frac{1}{2}) \left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{m+k-\frac{1}{2}} s^k U \left(m+k+\frac{1}{2}; m+k+\frac{1}{2}; \frac{s(\alpha-r)^2}{1-r^2} \right) \\
 &- \sum_{m=1}^M \sum_{k=0}^{m-1} c_k^{(m-1,2)} \Gamma(m+k+\frac{1}{2}) \left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{m+k-\frac{1}{2}} s^k U \left(m+k+\frac{1}{2}; m+k+\frac{1}{2}; \frac{s(\alpha-r)^2}{1-r^2} \right) \\
 &+ O \left(\left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{M+\frac{1}{2}} \Gamma(M+\frac{3}{2}) U \left(M+\frac{3}{2}; M+\frac{3}{2}; \frac{s(\alpha-r)^2}{1-r^2} \right) \right)
 \end{aligned}$$

where, $U(;;)$ is the Confluent Hypergeometric function in the notation of SLATER (1960), and, the $c_k^{(m,1)}$, $c_k^{(m-1,2)}$ are given by

$$(3.1.3) \quad \sum_{m=0}^{\infty} \sum_{k=0}^M c_k^{(m,1)} s^k t^{2(m+k)} = \frac{1}{2} [\chi(t) + \chi(-t)] \exp[st^2 + s \log(1-t^2)]$$

and,

$$(3.1.4) \quad \sum_{m=1}^{\infty} \sum_{k=0}^{m-1} c_k^{(m-1,2)} s^k t^{2(m+k-1)} = - \frac{i}{2\sqrt{1-r^2}} [\chi(t) - \chi(-t)] \exp[st^2 + s \log(1-t^2)].$$

Now by equations (2.6.7) and (2.6.8), we have, for $r < \alpha$,

$$(3.1.5) \quad H(r)$$

$$\begin{aligned} &= \frac{K'(1-r^2)^s}{2\pi} \sum_{m=0}^M \sum_{k=0}^m c_k^{(m,1)} \Gamma(m+k+\frac{1}{2}) \left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{m+k} s^k U\left(m+k+\frac{1}{2}; m+k+\frac{1}{2}; \frac{s(\alpha-r)^2}{1-r^2}\right) \\ &- \frac{K'(1-r^2)^{s+\frac{1}{2}}}{2\pi} \sum_{m=1}^M \sum_{k=0}^{m-1} c_k^{(m-1,2)} \Gamma(m+k+\frac{1}{2}) \left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{m+k-\frac{1}{2}} s^k U\left(m+k+\frac{1}{2}; m+k+\frac{1}{2}; \frac{s(\alpha-r)^2}{1-r^2}\right) \\ &+ O\left(\left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{M+\frac{1}{2}} \Gamma(M+\frac{3}{2}) U\left(M+\frac{3}{2}; M+\frac{3}{2}; \frac{s(\alpha-r)^2}{1-r^2}\right)\right), \end{aligned}$$

and for $r > \alpha$,

$$(3.1.6) \quad H(r)$$

$$\begin{aligned} &= 1 - \frac{K'(1-r^2)^s}{2\pi} \sum_{m=0}^M \sum_{k=0}^m c_k^{(m,1)} \Gamma(m+k+\frac{1}{2}) \left\{ \frac{(r-\alpha)^2}{1-r^2} \right\}^{m+k} s^k U\left(m+k+\frac{1}{2}; m+k+\frac{1}{2}; \frac{s(r-\alpha)^2}{1-r^2}\right) \\ &- \frac{K'(1-r^2)^{s+\frac{1}{2}}}{2\pi} \sum_{m=1}^M \sum_{k=0}^{m-1} c_k^{(m-1,2)} \Gamma(m+k+\frac{1}{2}) \left\{ \frac{(r-\alpha)^2}{1-r^2} \right\}^{m+k-\frac{1}{2}} s^k U\left(m+k+\frac{1}{2}; m+k+\frac{1}{2}; \frac{s(r-\alpha)^2}{1-r^2}\right) \\ &+ O\left(\left\{ \frac{(r-\alpha)^2}{1-r^2} \right\}^{M+\frac{1}{2}} \Gamma(M+\frac{3}{2}) U\left(M+\frac{3}{2}; M+\frac{3}{2}; \frac{s(r-\alpha)^2}{1-r^2}\right)\right). \end{aligned}$$

Now,

$$(3.1.7) \quad \Gamma(\sigma) \left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{\sigma-1} U\left(\sigma; \sigma; \frac{s(\alpha-r)^2}{1-r^2}\right) = \int_0^\infty \frac{v^{\sigma-1}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv] dv ,$$

and

$$(3.1.8) \quad U\left(\frac{1}{2}; \frac{1}{2}; \frac{s(\alpha-r)^2}{1-r^2}\right) = \exp \frac{s(\alpha-r)^2}{1-r^2} \int_{\frac{s(\alpha-r)^2}{1-r^2}}^\infty v^{-\frac{1}{2}} \exp[-v] dv .$$

Using (3.1.7) for $m+k > 0$ and (3.1.8) for $m+k = 0$, we get, in terms of the above integrals, for $r < \alpha$,

$$\begin{aligned} (3.1.9) \quad & H(r) \\ &= \frac{K'(1-r^2)^s}{2\pi} c_0^{(0,1)} \Gamma\left(\frac{1}{2}\right) \exp\left[\frac{s(\alpha-r)^2}{1-r^2}\right] \int_{\frac{s(\alpha-r)^2}{1-r^2}}^\infty v^{-\frac{1}{2}} \exp[-v] dv \\ &+ \frac{K'(1-r^2)^{s-\frac{1}{2}} (\alpha-r)}{2\pi} \sum_{m=1}^M \sum_{k=0}^m c_k^{(m,1)} s^k \left[\int_0^\infty \frac{v^{m+k-\frac{1}{2}}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv] dv \right] \\ &- \frac{K'(1-r^2)^{s+\frac{1}{2}}}{2\pi} \sum_{m=1}^M \sum_{k=0}^{m-1} c_k^{(m-1,2)} s^k \left[\int_0^\infty \frac{v^{m+k-\frac{1}{2}}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv] dv \right] \\ &+ 0 \left(\int_0^\infty \frac{v^{M+\frac{1}{2}}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv] dv \right) , \end{aligned}$$

and, for $r > \alpha$,

$$(3.1.10) \quad H(r)$$

$$\begin{aligned}
 &= 1 - \frac{K'(1-r^2)^s}{2\pi} c_o^{(0,1)} \Gamma\left(\frac{1}{2}\right) \exp\left[\frac{s(r-\alpha)^2}{1-r^2}\right] \int_{\frac{s(r-\alpha)^2}{1-r^2}}^{\infty} v^{-\frac{1}{2}} \exp[-v] dv \\
 &\quad - \frac{K'(1-r^2)^{s-\frac{1}{2}} (r-\alpha)}{2\pi} \sum_{m=1}^M \sum_{k=0}^m c_k^{(m,1)} s^k \left[\int_0^{\infty} \frac{v^{m+k-\frac{1}{2}}}{\left\{ \frac{(r-\alpha)^2}{1-r^2} + v \right\}} \exp[-sv] dv \right] \\
 &\quad - \frac{K'(1-r^2)^{s+\frac{1}{2}}}{2\pi} \sum_{m=1}^M \sum_{k=0}^{m-1} c_k^{(m-1,2)} s^k \left[\int_0^{\infty} \frac{v^{m+k-\frac{1}{2}}}{\left\{ \frac{(r-\alpha)^2}{1-r^2} + v \right\}} \exp[-sv] dv \right] \\
 &\quad + O\left(\int_0^{\infty} \frac{v^{M+\frac{1}{2}}}{\left\{ \frac{(r-\alpha)^2}{1-r^2} + v \right\}} \exp[-sv] dv \right) .
 \end{aligned}$$

Thus as $r \rightarrow \alpha$ (3.1.9) becomes

$$\begin{aligned}
 (3.1.11) \quad H(\alpha-0) &= \frac{K^*(1-\alpha^2)^s}{2} c_o^{*(0,1)} \\
 &\quad - \frac{K^*(1-\alpha^2)^{s+\frac{1}{2}}}{2\pi} \sum_{m=1}^M \sum_{k=0}^{m-1} c_k^{*(m-1,2)} \frac{\Gamma(m+k-\frac{1}{2})}{s^{m-\frac{1}{2}}} \\
 &\quad + O\left(\frac{1}{s^{M+\frac{1}{2}}} \right)
 \end{aligned}$$

and (3.1.10) becomes

$$(3.1.12) \quad H(\alpha+0) = 1 - \frac{K^*(1-\alpha^2)^s}{2} c_o^{*(0,1)}$$

$$- \frac{K^*(1-\alpha^2)^{s+\frac{1}{2}}}{2\pi} \sum_{m=1}^M \sum_{k=0}^{m-1} c_k^{*(m-1,2)} \frac{\Gamma(m+k-\frac{1}{2})}{s^{m-\frac{1}{2}}} + O\left(\frac{1}{s^{M+\frac{1}{2}}}\right)$$

where,

$$(3.1.13) \quad K^* = K' \quad \text{at} \quad r = \alpha.$$

$$(3.1.14) \quad c_k^{*(,)} = c_k^{(,)} \quad \text{at} \quad r = \alpha$$

and the interchange and limits and integration is justified since the integrals are uniformly convergent.

Now by (3.1.3)

$$(3.1.15) \quad c_o^{*(0,1)} = \chi^*(0),$$

where, by equation (2.6.6)

$$\chi^*(0) = g(\alpha)$$

and

$$(3.1.16) \quad \chi(t) = \chi^*(t) \quad \text{at} \quad r = \alpha.$$

By Appendix II, (equation 15)

$$(3.1.17) \quad \frac{K^*(1-\alpha^2)^s \chi^*(0)}{2} = \frac{K^*(1-\alpha^2)^s g(\alpha)}{2} = \frac{1}{2}.$$

Thus,

$$(3.1.18) \quad H(\alpha-0) = H(\alpha+0) = H(\alpha)$$

$$= \frac{1}{2} - \frac{K^*(1-\alpha^2)^{s+\frac{1}{2}}}{2\pi} \sum_{m=1}^M \sum_{k=0}^{m-1} c_k^{*(m-1,2)} \frac{\Gamma(m+k-\frac{1}{2})}{s^{m-\frac{1}{2}}} \\ + O\left(\frac{1}{s^{M+\frac{1}{2}}}\right).$$

We take (3.1.18) to be the asymptotic value of $H(r)$ at $r = \alpha$. It may be noted that the expansion (3.1.18) may also be derived independently, directly from the integral, by the method outlined in ERDELYI and WYMAN (1963).

Thus in equations (3.1.9) and (3.1.10), for $M = 0$, we have the following asymptotic expressions, for $r < \alpha$

$$(3.1.19) \quad H(r) = \frac{K^*(1-r^2)^s}{2\pi} \chi(0) \Gamma(\tfrac{1}{2}) \exp\left[\frac{s(\alpha-r)^2}{1-r^2}\right] \int_{\frac{s(\alpha-r)^2}{1-r^2}}^{\infty} v^{-\frac{1}{2}} \exp[-v] dv \\ + O\left(\int_0^{\infty} \frac{v^{\frac{1}{2}}}{\left\{\frac{(\alpha-r)^2}{1-r^2} + v\right\}} \exp[-sv] dv\right)$$

for $r > \alpha$

$$(3.1.20) \quad H(r) = 1 - \frac{K^*(1-r^2)^s}{2\pi} \chi(0) \Gamma(\tfrac{1}{2}) \exp\left[\frac{s(r-\alpha)^2}{1-r^2}\right] \int_{\frac{s(r-\alpha)^2}{1-r^2}}^{\infty} v^{-\frac{1}{2}} \exp[-v] dv \\ + O\left(\int_0^{\infty} \frac{v^{\frac{1}{2}}}{\left\{\frac{(r-\alpha)^2}{1-r^2} + v\right\}} \exp[-sv] dv\right)$$

which, for $r = \alpha$, give

$$(3.1.21) \quad H(\alpha) = \frac{1}{2} + O\left(\frac{1}{s^2}\right)$$

which would be an approximation to the first term in (3.1.18). Thus, we take as the asymptotic representations for $r \leq \alpha$,

$$(3.1.22) \quad H(r) \sim \frac{K'(1-r^2)^s}{2\pi} \chi(0) \Gamma\left(\frac{1}{2}\right) \exp\left[\frac{s(\alpha-r)^2}{1-r^2}\right] \int_{\frac{s(\alpha-r)^2}{1-r^2}}^{\infty} v^{-\frac{1}{2}} \exp[-v] dv$$

and, for $r \geq \alpha$,

$$(3.1.23) \quad H(r) \sim 1 - \frac{K'(1-r^2)^s}{2\pi} \chi(0) \Gamma\left(\frac{1}{2}\right) \exp\left[\frac{s(r-\alpha)^2}{1-r^2}\right] \int_{\frac{s(r-\alpha)^2}{1-r^2}}^{\infty} v^{-\frac{1}{2}} \exp[-v] dv.$$

By equation (2.6.6),

$$(3.1.24) \quad \chi(0) = g(r).$$

Thus corresponding to equations (2.5.7) and (2.5.8), we have for $r \leq \alpha$,

$$(3.1.25) \quad H(r) \sim \frac{K'(1-r^2)^s g(r)}{2\pi} \Gamma\left(\frac{1}{2}\right) \exp\left[\frac{s(\alpha-r)^2}{1-r^2}\right] \int_{\frac{s(\alpha-r)^2}{1-r^2}}^{\infty} v^{-\frac{1}{2}} \exp[-v] dv,$$

and for $r \geq \alpha$,

$$(3.1.26) \quad H(r) \sim 1 - \frac{K'(1-r^2)^s g(r)}{2\pi} \Gamma\left(\frac{1}{2}\right) \exp\left[\frac{s(r-\alpha)^2}{1-r^2}\right] \int_{\frac{s(r-\alpha)^2}{1-r^2}}^{\infty} v^{-\frac{1}{2}} \exp[-v] dv.$$

§ 2. Approximations to Distribution Functions of two Sample Serial Correlation Coefficients.

By equations (2.4.6), (2.5.7) and (2.5.8), the z-integral has form

$$(3.2.1) \quad \frac{K'}{2\pi i} \int_{\bar{A}}^A \frac{1}{z-\alpha} g(z) (1-2rz+z^2)^s dz$$

for both the circular and non-circular cases.

(i) The Circular Case.

By equation (2.3.20), the correspondence with (3.2.1) is

$$(3.2.2) \quad (i) \quad s = \frac{n}{2} - 1$$

$$(ii) \quad K' = \frac{1}{(1-2\alpha r + \alpha^2)^{\frac{n}{2} - 1}}$$

$$(iii) \quad g(z) = \frac{1-z^2}{1-\alpha z} \quad .$$

Thus by equations (3.1.25) and (3.1.26), for $r \leq \alpha$

$$(3.2.3) \quad H(r) \sim \frac{1}{2\sqrt{\pi}} \frac{(1-r^2)^{\frac{n}{2}}}{(1-2\alpha r + \alpha^2)^{\frac{n}{2} - 1}} \frac{\exp\left[\left(\frac{n}{2} - 1\right) \frac{(\alpha-r)^2}{1-r^2}\right]}{(1-\alpha r)} \\ \times \int_0^\infty v^{-\frac{1}{2}} \exp[-v] dv \left\{ \frac{n}{2} - 1 \right\} \frac{(\alpha-r)^2}{1-r^2}$$

and, for $r \geq \alpha$

$$(3.2.4) \quad H(r) \sim 1 - \frac{1}{2\sqrt{\pi}} \frac{(1-r^2)^{\frac{n}{2}}}{(1-2\alpha r + \alpha^2)^{\frac{n}{2}-1}} \frac{\exp \left[\left(\frac{n}{2} - 1 \right) \frac{(r-\alpha)^2}{1-r^2} \right]}{(1-\alpha r)} \\ \times \int_0^\infty v^{-\frac{1}{2}} \exp[-v] dv \\ \left\{ \left(\frac{n}{2} - 1 \right) \frac{(r-\alpha)^2}{1-r^2} \right\}$$

and, in particular, for $r = \alpha$,

$$(3.2.5) \quad H(\alpha) \sim \frac{1}{2}.$$

(ii) The Non-circular Case.

By equation (2.3.21), the correspondence with (3.2.1) is

$$(3.2.6) \quad (i) \quad s = \frac{n}{2} - 1 \\ (ii) \quad K' = \frac{(1-\alpha^2)^{\frac{1}{2}}}{(1-2\alpha r + \alpha^2)^{\frac{n}{2}-2}} \\ (iii) \quad g(z) = \frac{(1-z^2)^{\frac{3}{2}}}{(1-\alpha z)^2 (1-\alpha r + \alpha z - rz)}.$$

Thus, by equations (3.1.25) and (3.1.26), for $r \leq \alpha$,

$$(3.2.7) \quad H(r) \sim \frac{1}{2\sqrt{\pi}} \frac{(1-\alpha^2)^{\frac{1}{2}} (1-r^2)^{\frac{n}{2}-\frac{1}{2}}}{(1-\alpha r)^2 (1-2\alpha r + \alpha^2)^{\frac{n}{2}-2}} \exp \left[\left(\frac{n}{2} - 1 \right) \frac{(\alpha-r)^2}{1-r^2} \right] \\ \times \int_0^\infty v^{-\frac{1}{2}} \exp[-v] dv \\ \left\{ \left(\frac{n}{2} - 1 \right) \frac{(\alpha-r)^2}{1-r^2} \right\}$$

and for $r \geq \alpha$

$$(3.2.8) \quad H(r) \sim 1 - \frac{1}{2\sqrt{\pi}} \frac{(1-\alpha^2)^{\frac{1}{2}} (1-r^2)^{\frac{n}{2}-\frac{1}{2}}}{(1-\alpha r)^2 (1-2\alpha r + \alpha^2)^{\frac{n}{2}-2}} \exp\left[\left(\frac{n}{2}-1\right) \frac{(r-\alpha)^2}{1-r^2}\right]$$

$$\times \int_0^\infty v^{-\frac{1}{2}} \exp[-v] dv, \\ \left\{ \left(\frac{n}{2}-1\right) \frac{(r-\alpha)^2}{1-r^2} \right\}$$

and, in particular, for $r = \alpha$,

$$(3.2.9) \quad H(\alpha) \sim \frac{1}{2}.$$

The magnitude of error involved in making the above approximations, for both the circular and non-circular cases, is

$$(3.2.10) \quad R = O\left(\frac{1}{n^{\frac{1}{2}}}\right) \quad \text{as } n \rightarrow \infty.$$

This order relation is uniform in r and n . A more exact relation uniform in n and not in r may be derived from the equations (3.1.19) and (3.1.20).

Now, for $r \leq \alpha$, consider

$$(3.2.11) \quad \exp\left[\left(\frac{n}{2}-1\right) \frac{(\alpha-r)^2}{1-r^2}\right] \int_0^\infty v^{-\frac{1}{2}} \exp[-v] dv, \\ \left\{ \left(\frac{n}{2}-1\right) \frac{(\alpha-r)^2}{1-r^2} \right\}$$

Put $v = \frac{1}{2} t^2.$

Then,

$$dv = t \, dt \, ,$$

$$v^{-\frac{1}{2}} = \sqrt{2} / t$$

and,

$$v = \left(\frac{n}{2} - 1\right) \frac{(\alpha - r)^2}{1 - r^2} \text{ corresponds to } t = \frac{\sqrt{n-2} (\alpha - r)}{\sqrt{1 - r^2}} \, .$$

Thus, (3.2.11) becomes

$$(3.2.12) \quad \sqrt{2} \int_0^\infty \exp\left[-\frac{1}{2}t^2\right] dt \exp\left[-\frac{1}{2} \left\{ \frac{\sqrt{n-2} (\alpha - r)}{\sqrt{1 - r^2}} \right\}^2\right]$$

$$= \sqrt{2} \int_{-\infty}^{\left\{ \frac{-\sqrt{n-2} (\alpha - r)}{\sqrt{1 - r^2}} \right\}} \exp\left[-\frac{1}{2}t^2\right] dt \exp\left[-\frac{1}{2} \left\{ \frac{\sqrt{n-2} (\alpha - r)}{\sqrt{1 - r^2}} \right\}^2\right]$$

$$= \sqrt{2} \frac{\Phi \left[\frac{-\sqrt{n-2} (\alpha - r)}{\sqrt{1 - r^2}} \right]}{\phi \left[\frac{-\sqrt{n-2} (\alpha - r)}{\sqrt{1 - r^2}} \right]}$$

where $\Phi()$ is the standardized normal distribution function and $\phi()$ the standard normal probability density function.

The corresponding result for $r \geq \alpha$ is

$$(3.2.13) \quad \sqrt{2} \frac{\Phi \left[\frac{-\sqrt{n-2} (r - \alpha)}{\sqrt{1 - r^2}} \right]}{\phi \left[\frac{-\sqrt{n-2} (r - \alpha)}{\sqrt{1 - r^2}} \right]} \, .$$

Thus, for the circular case, we have by equations (3.2.3) and (3.2.4),
for $r \leq \alpha$,

$$(3.2.14) \quad H(r) \sim \frac{1}{\sqrt{2\pi}} \frac{(1-r^2)^{\frac{n}{2}}}{(1-\alpha r)(1-2\alpha r+\alpha^2)^{\frac{n}{2}-1}} \frac{\Phi \left[\frac{-\sqrt{n-2} |\alpha-r|}{\sqrt{1-r^2}} \right]}{\Phi \left[\frac{-\sqrt{n-2} |\alpha-r|}{\sqrt{1-r^2}} \right]}$$

and, for $r \geq \alpha$

$$(3.2.15) \quad H(r) \sim 1 - \frac{1}{\sqrt{2\pi}} \frac{(1-r^2)^{\frac{n}{2}}}{(1-\alpha r)(1-2\alpha r+\alpha^2)^{\frac{n}{2}-1}} \frac{\Phi \left[\frac{-\sqrt{n-2} |\alpha-r|}{\sqrt{1-r^2}} \right]}{\Phi \left[\frac{-\sqrt{n-2} |\alpha-r|}{\sqrt{1-r^2}} \right]} .$$

For the non-circular case, we have by equations (3.2.7) and (3.2.8), for $r \leq \alpha$

$$(3.2.16) \quad H(r) \sim \frac{1}{\sqrt{2\pi}} \frac{(1-\alpha)^{\frac{1}{2}} (1-r^2)^{\frac{n}{2}-\frac{1}{2}}}{(1-\alpha r)^2 (1-2\alpha r+\alpha^2)^{\frac{n}{2}-2}} \frac{\Phi \left[\frac{-\sqrt{n-2} |\alpha-r|}{\sqrt{1-r^2}} \right]}{\Phi \left[\frac{-\sqrt{n-2} |\alpha-r|}{\sqrt{1-r^2}} \right]}$$

and, for $r \geq \alpha$

$$(3.2.17) \quad H(r) \sim 1 - \frac{1}{\sqrt{2\pi}} \frac{(1-\alpha^2)^{\frac{1}{2}} (1-r^2)^{\frac{n}{2}-\frac{1}{2}}}{(1-\alpha r)^2 (1-2\alpha r+\alpha^2)^{\frac{n}{2}-2}} \frac{\Phi \left[\frac{-\sqrt{n-2} |\alpha-r|}{\sqrt{1-r^2}} \right]}{\Phi \left[\frac{-\sqrt{n-2} |\alpha-r|}{\sqrt{1-r^2}} \right]} .$$

As discussed earlier, in equation (3.2.10), the order of the magnitude of the error involved in making the above approximations is

$$(3.2.18) \quad R = O\left(n^{-\frac{1}{2}}\right), \quad n \rightarrow \infty,$$

uniformly in n and r .

These results, namely, (3.2.14), (3.2.15), (3.2.16) and (3.2.17), are represented graphically in Appendix four. The order of the magnitude of error has not been taken into account in the tracings of the curves.

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APPENDIX I

COMPLEX INTEGRALS IN THE SENSE OF A CAUCHY PRINCIPLE VALUE

Assumptions:

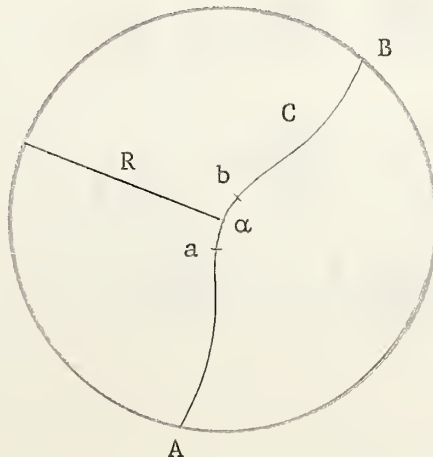
- (1) $f(z)$ is a function defined in some region of the z -plane.
- (2) Consider a fixed point $z = \alpha$, and for some $R > 0$ (fixed), consider a neighbourhood $|z - \alpha| \leq R$. $f(z)$ is taken to be regular for all points of $|z - \alpha| \leq R$ except at $z = \alpha$.
- (3) The point $z = \alpha$ is a simple pole of $f(z)$.
- (4) C is any continuous curve in the z -plane passing through $z = \alpha$ and intersecting with $|z - \alpha| = R$ at $z = A$ and $z = B$.

We wish to determine sufficient conditions under which the integral

$$\lim_{\substack{a \rightarrow \alpha \\ b \rightarrow \alpha}} \left[\int_{[C]}^a + \int_b^B \right] f(z) dz \quad (1)$$

exists in the sense of a Cauchy Principle Value, where the limits $a \rightarrow \alpha$, $b \rightarrow \alpha$ are taken to be limits through the set C .

The configuration is as in the diagram.



By assumption

$$f(z) = \frac{K}{z-\alpha} + g(z)$$

for all z in $|z-\alpha| \leq R$ where $g(z)$ is regular for all z in $|z-\alpha| \leq R$, and K is constant with respect to z . Thus,

$$\begin{aligned} & \lim_{\substack{a \rightarrow \alpha \\ b \rightarrow \alpha}} \left(\int_A^a + \int_b^B \right) f(z) dz \\ &= \lim_{\substack{a \rightarrow \alpha \\ b \rightarrow \alpha}} \left(\int_A^a + \int_b^B \right) \left[\frac{K}{z-\alpha} + g(z) \right] dz \\ &= \lim_{\substack{a \rightarrow \alpha \\ b \rightarrow \alpha}} \left(\int_A^a + \int_b^B \right) \frac{K}{z-\alpha} dz + \lim_{\substack{a \rightarrow \alpha \\ b \rightarrow \alpha}} \left(\int_A^a + \int_b^B \right) g(z) dz . \end{aligned}$$

Now, by regularity of $g(z)$ for all z in $|z-\alpha| \leq R$,

$$\lim_{\substack{a \rightarrow \alpha \\ b \rightarrow \alpha}} \left(\int_A^a + \int_b^B \right) g(z) dz = \int_A^B g(z) dz ,$$

and, further,

$$\begin{aligned} & \lim_{\substack{a \rightarrow \alpha \\ b \rightarrow \alpha}} \left(\int_A^a + \int_b^B \right) \frac{K}{z-\alpha} dz \\ &= K \lim_{a \rightarrow \alpha} \left[\log \left\{ \frac{a-\alpha}{A-\alpha} \cdot \frac{B-\alpha}{b-\alpha} \right\} \right] \\ &= K \lim_{\substack{a \rightarrow \alpha \\ b \rightarrow \alpha}} \log \left(\frac{B-\alpha}{A-\alpha} \right) + K \lim_{\substack{a \rightarrow \alpha \\ b \rightarrow \alpha}} \left(\frac{a-\alpha}{b-\alpha} \right) \\ &= K \log \left(\frac{B-\alpha}{A-\alpha} \right) + K \lim_{\substack{a \rightarrow \alpha \\ b \rightarrow \alpha}} \left(\frac{a-\alpha}{b-\alpha} \right) . \end{aligned}$$

Thus

$$\lim_{\substack{a \rightarrow \alpha \\ b \rightarrow \alpha}} \left(\int_A^a + \int_b^B \right) f(z) dz$$

$$= \lim_{\substack{a \rightarrow \alpha \\ b \rightarrow \alpha}} K \log \left(\frac{a-\alpha}{b-\alpha} \right) + K \log \left(\frac{B-\alpha}{A-\alpha} \right) + \int_A^B g(z) dz$$

where the integral on the left will exist if and only if

$$\lim_{\substack{a \rightarrow \alpha \\ b \rightarrow \alpha}} K \log \left(\frac{a-\alpha}{b-\alpha} \right) \text{ is finite.}$$

This is a sufficient condition that (1) exist as a Cauchy Principle Value.

This above condition may be transformed to something much simpler in form.

Consider the normal to the curve C at $z = \alpha$, and consider the transformation

$$z - \alpha = t e^{i\theta}$$

where z, t and α are complex variables, θ is a fixed angle determined so that the normal to the curve C at $z = \alpha$ in the z -plane is transformed to be the line $\text{Im } t = 0$ in the t -plane. Thus in the t -plane, the limit becomes

$$\lim_{\substack{t_a \rightarrow 0 \\ t_b \rightarrow 0}} K \log \frac{t_a}{t_b}$$

where t_a, t_b are the t -images of a and b respectively, and the limits are through the set C_t (the t -image of C in the z -plane).

Now we consider a special case of interest. Assume that C_t , the t -image of the curve C in the z -plane is symmetric about $\text{Im } t = 0$, and that $t_a = \bar{t}_b$ (the complex conjugate of t_b).

$$\begin{aligned} \text{Thus if } t_b &= |t_b| e^{i\varphi} \\ t_a &= |t_b| e^{-i\varphi} \end{aligned} \quad 0 \leq \varphi \leq 2\pi$$

and

$$\begin{aligned} \lim_{\substack{t_a \rightarrow 0 \\ t_b \rightarrow 0}} \quad & \text{becomes} \quad \lim_{\varphi \rightarrow \Phi} . \end{aligned}$$

Thus we have

$$\begin{aligned} \lim_{\substack{t_a \rightarrow 0 \\ t_b \rightarrow 0}} K \log \left(t_a / t_b \right) &= \lim_{\varphi \rightarrow \Phi} K \log \left(e^{-i\varphi} / e^{i\varphi} \right) \\ &= \lim_{\varphi \rightarrow \Phi} K \log \left(e^{-2i\varphi} \right) \\ &= \lim_{\varphi \rightarrow \Phi} -2iK\varphi \\ &= -2iK\Phi \end{aligned}$$

which is constant and finite.

[cf. SANSONE and GERRETSEN (1960) pp. 127-128].

APPENDIX II

CERTAIN EQUALITIES

Equations (2.3.18) and (2.3.19) are of the form

$$(1) \quad \frac{\varphi(\zeta(z)) \zeta'(z)}{\zeta(z)} = \frac{K}{z-\alpha} h(z) (1-2rz+z^2)^s$$

where s is defined as in equation (2.4.3).

In equation (2.3.18)

$$K = \frac{1 - \alpha^n}{(1-2\alpha r + \alpha^2)^{\frac{n}{2} - 1}}$$

and

$$h(z) = \frac{1-z^2}{(1-\alpha z)(1-z^n)},$$

while in equation (2.3.19),

$$K = \frac{(1-\alpha^2)^{\frac{1}{2}}}{(1-2\alpha r + \alpha^2)^{\frac{n}{2} - 2}}$$

and

$$h(z) = \frac{(1-z^2)^{\frac{3}{2}}}{(1-\alpha z) \{ (1-\alpha z)^2 (1-\alpha r + \alpha z - rz)^2 + z^{2n-2} (\alpha - z)^2 (\alpha - r + z - \alpha rz)^2 \}^{\frac{1}{2}}},$$

and in both equations (2.3.18) and (2.3.19)

$$(2) \quad \frac{\zeta'(z)}{\zeta(z)} = \frac{(1-z^2)(1-2\alpha r + \alpha^2)}{(z-\alpha)(1-\alpha z)(1-2rz+z^2)}.$$

Thus,

$$(3) \quad K h(z) (1-2rz+z^2)^s = \varphi(\zeta(z)) \frac{(1-z^2)(1-2\alpha r+\alpha^2)}{(1-\alpha z)(1-2rz+z^2)}$$

and,

$$(4) \quad K h(\alpha) (1-2\alpha r+\alpha^2)^s = \lim_{z \rightarrow \alpha} K h(z) (1-2rz+z^2)^s$$

$$= \varphi(\zeta(\alpha)) \frac{(1-\alpha^2)(1-2\alpha r+\alpha^2)}{(1-\alpha^2)(1-2\alpha r+\alpha^2)} = \varphi(\zeta(\alpha)) .$$

Now,

$$(5) \quad \zeta(z) = i\alpha - \frac{iz(1-2\alpha r+\alpha^2)}{1-2rz+z^2} ,$$

so that

$$(6) \quad \zeta(\alpha) = i\alpha - \frac{i\alpha(1-2\alpha r+\alpha^2)}{1-2\alpha r+\alpha^2}$$

$$= i\alpha - i\alpha = 0 .$$

Thus,

$$(7) \quad K h(\alpha) (1-2\alpha r+\alpha^2)^s = \varphi(0) ,$$

and since $\varphi(\)$ is a characteristic function

$$(8) \quad \varphi(0) = 1.$$

Thus,

$$(9) \quad K h(\alpha) (1-2\alpha r+\alpha^2)^s = 1 .$$

Comparing (2.3.20) with (2.4.3) we see that

$$\begin{aligned}
 (10) \quad K' g(z) (1-2rz+z^2)^s &= \frac{1}{(1-2\alpha r+\alpha^2)^{\frac{n}{2}-1}} \frac{1-z^2}{1-\alpha z} (1-2rz+z^2)^s \\
 &= \frac{1-z^n}{1-\alpha^n} \frac{1-\alpha^n}{(1-2\alpha r+\alpha^2)^{\frac{n}{2}-1}} \frac{1-z^2}{(1-\alpha z)(1-z^n)} (1-2rz+z^2)^s \\
 &= \frac{1-z^n}{1-\alpha^n} K h(z) (1-2rz+z^2)^s .
 \end{aligned}$$

Thus,

$$\begin{aligned}
 (11) \quad K' g(\alpha) (1-2\alpha r+\alpha^2)^s &= \frac{1-\alpha^n}{1-\alpha^n} K h(\alpha) (1-2\alpha r+\alpha^2)^s \\
 &= 1 .
 \end{aligned}$$

Again, comparing (2.3.21) with (2.4.3), we have

$$\begin{aligned}
 (12) \quad K' g(z) (1-2rz+z^2)^s &= \frac{(1-\alpha^2)^{\frac{1}{2}}}{(1-2\alpha r+\alpha^2)^{\frac{n}{2}-2}} \frac{(1-z^2)^{\frac{3}{2}}}{(1-\alpha z)^2(1-\alpha r+\alpha z-rz)} (1-2rz+z^2)^s \\
 &= \frac{\{(1-\alpha z)^2(1-\alpha r+\alpha z-rz)^2+z^{2n-2}(\alpha-z)^2(\alpha-r+z-\alpha rz)^2\}^{\frac{1}{2}}}{(1-\alpha z)(1-\alpha r+\alpha z-rz)} \\
 &\times \frac{(1-\alpha^2)^{\frac{1}{2}}}{(1-2\alpha r+\alpha^2)^{\frac{n}{2}-2}} \frac{(1-z^2)^{\frac{3}{2}}}{(1-\alpha z)\{(1-\alpha z)^2(1-\alpha r+\alpha z-rz)^2+z^{2n-2}(\alpha-z)^2(\alpha-r+z-\alpha rz)^2\}^{\frac{1}{2}}}
 \end{aligned}$$

$$= \frac{\{ (1-\alpha z)^2 (1-\alpha r + \alpha z - rz)^2 + z^{2n-1} (\alpha - z)^2 (\alpha - r + z - \alpha rz)^2 \}^{\frac{1}{2}}}{(1-\alpha z) (1-\alpha r + \alpha z - rz)}$$

$$K \ h(z) (1-2rz+z^2)^s .$$

Thus

$$\begin{aligned} (13) \quad K' \ g(\alpha) (1-2\alpha r + \alpha^2)^s &= \frac{(1-\alpha^2)(1-2\alpha r + \alpha^2)}{(1-\alpha^2)(1-2\alpha r + \alpha^2)} K \ h(\alpha) (1-2\alpha r + \alpha^2)^s \\ &= 1 , \end{aligned}$$

which justifies the statement of equation (2.5.4(a)). Thus, from

$$(14) \quad K' \ g(\alpha) (1-2\alpha r + \alpha^2)^s = 1$$

we conclude that

$$\begin{aligned} (15) \quad \lim_{r \rightarrow \alpha} K' \ g(\alpha) (1-2\alpha r + \alpha^2)^s &= K^* \ g^*(\alpha) (1-\alpha^2)^s \\ &= 1 \end{aligned}$$

where

$$K' = K^* \quad \text{for } r = \alpha$$

and

$$g(z) = g^*(z) \quad \text{for } r = \alpha .$$

APPENDIX III

AN ASYMPTOTIC EXPANSION.

In this appendix we give the details of the derivation of the asymptotic expansion for the integral

$$(1) \quad \int_{-1}^{+1} \frac{\chi(t)}{\alpha - r - i\sqrt{1-r^2} t} (1-t^2)^s dt, \quad [L^*]$$

with the assumptions (3.1.1) (i) - (iv).

Thus consider

$$\begin{aligned} (2) \quad I(r) &= \int_{-1}^{+1} \frac{\chi(t)}{\alpha - r - i\sqrt{1-r^2} t} (1-t^2)^s dt \\ &= \frac{1}{\sqrt{1-r^2}} \int_{-1}^{+1} \left\{ \frac{\chi(t)}{\sqrt{1-r^2}} - it \right\} (1-t^2)^s dt \\ &= \frac{1}{\sqrt{1-r^2}} \int_{-1}^{+1} \frac{\chi(t) \left\{ \frac{\alpha-r}{\sqrt{1-r^2}} + it \right\}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + t^2 \right\}} (1-t^2)^s dt \\ &= \frac{1}{\sqrt{1-r^2}} \left[\int_0^{+1} \frac{\chi(t) \left\{ \frac{\alpha-r}{\sqrt{1-r^2}} + it \right\}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + t^2 \right\}} (1-t^2)^s dt \right. \\ &\quad \left. + \int_0^{+1} \frac{\chi(-t) \left\{ \frac{\alpha-r}{\sqrt{1-r^2}} - it \right\}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + t^2 \right\}} (1-t^2)^s dt \right] \end{aligned}$$

$$= \frac{1}{\sqrt{1-r^2}} \left[\frac{\alpha-r}{\sqrt{1-r^2}} \int_{[L^*]}^1 \frac{[\chi(t) + \chi(-t)]}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + t^2 \right\}} (1-t^2)^s dt \right. \\ \left. + i \int_{[L^*]}^1 \frac{t[\chi(t) - \chi(-t)]}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + t^2 \right\}} (1-t^2)^s dt \right] ,$$

each integral being absolutely convergent.

Referring to equation (2.6.6), we have,

$$(3) \quad g(z) = g(r + i\sqrt{1-r^2} t) = \chi(t) ,$$

where $g(z)$ is an algebraic function of z , which is real valued for all real values of z , and is regular in some neighbourhood of $z = r$. Thus for some $R > 0$ and $|z-r| \leq \sqrt{1-r^2} R$

$$(4) \quad g(z) = \sum_{m=0}^{\infty} a_m (z-r)^m = \sum_{m=0}^{\infty} a_m (i\sqrt{1-r^2})^m t^m = \chi(t) ,$$

where the a_m are real. Thus,

$$(5) \quad \chi(t) + \chi(-t) = 2 \sum_{m=0}^{\infty} (-1)^m a_{2m} (1-r^2)^m t^{2m}$$

and

$$\begin{aligned}
 (6) \quad t[\chi(t) - \chi(-t)] &= 2i \sum_{m=0}^{\infty} (-1)^m a_{2m+1} (1-r^2)^{m+\frac{1}{2}} t^{2m+2} \\
 &= 2i\sqrt{1-r^2} t^2 \sum_{m=0}^{\infty} (-1)^m a_{2m+1} (1-r^2)^m t^{2m} .
 \end{aligned}$$

Put

$$(7) \quad \chi(t) + \chi(-t) = 2 f_1(t^2) ,$$

and

$$(8) \quad t[\chi(t) - \chi(-t)] = 2i\sqrt{1-r^2} t^2 f_2(t^2)$$

which means that for $|t| \leq R$

$$(9) \quad f_1(t^2) = \sum_{m=0}^{\infty} (-1)^m a_{2m} (1-r^2)^m (t^2)^m$$

and

$$(10) \quad f_2(t^2) = \sum_{m=0}^{\infty} (-1)^m a_{2m+1} (1-r^2)^m (t^2)^m$$

where,

$$(11) \quad a_m = \frac{d^m}{dz^m} g(z) \Big|_{z=r} \Big/ m! \quad \text{is always real.}$$

Thus (2) becomes

$$\begin{aligned}
 (12) \quad I(r) &= \frac{2(\alpha-r)}{1-r^2} \int_{[L^*]}^1 \frac{f_1(t^2)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + t^2 \right\}} (1-t^2)^s dt \\
 &\quad - 2 \int_{[L^*]}^1 \frac{t^2 f_2(t^2)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + t^2 \right\}} (1-t^2)^s dt \\
 &= \frac{\alpha-r}{1-r^2} \int_{[L^*]}^1 \frac{t^{-1} f_1(t^2)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + t^2 \right\}} (1-t^2)^s d(t^2) \\
 &\quad - \int_{[L^*]}^1 \frac{t f_2(t^2)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + t^2 \right\}} (1-t^2)^s d(t^2) \\
 &= \frac{\alpha-r}{1-r^2} \int_{[D]}^1 \frac{v^{-\frac{1}{2}} f_1(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} (1-v)^s dv \\
 &\quad - \int_{[D]}^1 \frac{v^{\frac{1}{2}} f_2(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} (1-v)^s dv ,
 \end{aligned}$$

where,

$$(13) \quad v = t^2$$

and D is the v-image of L^* .

Now consider

$$(14) \quad (1-v)^s = \exp[s \log(1-v)]$$

$$= \exp[-sv \left\{ -\frac{\log(1-v)}{v} \right\}]$$

and let

$$(15) \quad -\frac{\log(1-v)}{v} = \psi(v) .$$

Now

$$(16) \quad \psi(v) = -\frac{\log(1-v)}{v} = \sum_{k=0}^{\infty} \frac{v^k}{k+1} , \quad (|v| < 1) .$$

Thus $\psi(v)$ is regular at $v = 0$, and thus is regular for all v in $[0,1)$.

Thus (12) becomes,

$$(17) \quad I(r) = \frac{\alpha-r}{1-r^2} \int_{[D]}^1 \frac{v^{-\frac{1}{2}} f_1(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv \psi(v)] dv$$

$$- \int_{[D]}^1 \frac{v^{\frac{1}{2}} f_2(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv \psi(v)] dv ,$$

where,

(i) $f_i(v)$, $i = 1, 2$, and $\psi(v)$ are regular in a neighbourhood of $v = 0$,

- (ii) $\operatorname{Re}(sv \psi(v)) > 0$ for all v in $(0,1]$. [Indeed $sv \psi(v)$ is real valued and monotonically increasing in $[0,1]$, which is a path of "steepest descent"],
- (iii) for some fixed $v_0 \neq 0$, and v_0 in $[0,1)$, there is a fixed $\delta > 0$, such that

$$|\psi(v)| \geq \delta$$

for all v in $[v_0,1]$,

- (iv) the integrals are absolutely convergent.

Thus we concern ourselves with deriving the asymptotic expansion of an integral of the form

$$(18) \quad I^* = \int_{\substack{0 \\ [D]}}^1 \frac{v^{\lambda-1} f(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv \psi(v)] dv$$

under the conditions (17), (i) - (iv).

From ERDELYI and WYMAN (1963) we have the following:

Assume that the integrand has the form

$$(19) \quad H = v^{p\lambda-1} f^*(v) \exp[-sv^p \psi(v)]$$

with the following assumptions;

- (20) (i) The v -plane is cut along $\arg v = \gamma$ from $v = 0$ to $v = \infty e^{i\gamma}$, $\gamma \leq \arg v \leq \gamma + 2\pi$.

(ii) p is a fixed real number and $p > 0$.

(iii) λ is always a fixed, finite, complex number. When $\operatorname{Re} \lambda > 0$, $p > 0$ suffices, when λ unrestricted it is necessary that $p \geq 1$.

(iv) $f^*(v)$ and $\psi(v)$ are both analytic on the path of integration, say D , and for some fixed $R > 0$, are both regular in $|v| \leq R$.

(v) s is a complex number $\rightarrow \infty$ through the closed unbounded point-set, $S(\Delta)$, of the s -plane, where $S(\Delta)$ is defined by

$$(4\ell-1) \frac{\pi}{2} - p\gamma + \Delta \leq \operatorname{Arg} s \leq (4\ell+1) \frac{\pi}{2} - p\gamma - \Delta$$

where Δ is fixed, $0 < \Delta \leq \pi$.

(vi) For all paths of integration, D , used $\int_{[D]} Hdv$ exists as an absolutely convergent integral for each sufficiently large s in $S(\Delta)$.

Under these conditions the following theorem holds.

Theorem I:

Let D_1 be a continuous curve joining the fixed, finite, point $v = v_0 \neq 0$ to the fixed point $v = B$, where B might or might not be the point at infinity. Further suppose that the points of D_1 are contained

in the point-set $|v| \geq |v_0|$. If

$$(21) \quad H = v^{p\lambda-1} f^*(v) \exp[-sv^p \psi(v)]$$

satisfies

- (i) $|\psi(v)| \geq \delta_1 > 0$, for some fixed δ_1 , uniformly for points of D_1 .
- (ii) for $\tau(s,v) = \text{Arg}(sv^p \psi(v))$, $\cos \tau \geq \delta_2 > 0$, for some fixed δ_2 , uniformly in (s,v) , where s is sufficiently large in $S(\Delta)$ and v belongs to D_1 .
- (iii) $\int_{[D_1]} H dv$ exists as an absolutely convergent integral for each, fixed, sufficiently large s in $S(\Delta)$,

then

$$(22) \quad \int_{[D_1]} H dv \approx 0$$

uniformly in s , as $s \rightarrow \infty$ in $S(\Delta)$.

The order relation implied in (22) $\left[\text{i.e. } \int_{[D]} H dv = o\left(\frac{1}{s^{\lambda + \frac{n}{p}}}\right) \right]$

for $n = 0, 1, \dots$] is uniform in s only, since there is no assumption that the uniformity extends to any parameters that may occur in the integral. [cf. ERDÉLYI and WYMAN (1963) and (1963a)] .

In this appendix, the following restrictions are maintained with reference to 20,(i)- (vi) .

(23) (i) the v -plane is cut along $\arg v = 0$

(ii) $p = 1$

(iii) λ is real and always $\lambda > 0$

(iv) the same as in (20)

(v) s is a real variable $\rightarrow \infty$

(vi) all paths of integration are along the line $\text{Im } v = 0$,
from $v = 0$ to $v = B$, where B might or might not be
the point at infinity.

Digression on the Confluent Hypergeometric function.

We adopt the notation of SLATER (1960). All variables and parameters (except perhaps the variable of integration) are assumed real. Thus for $a > 0$, $x > 0$

$$(24) \quad U(a ; b ; x) = \frac{x^{1-b}}{\Gamma(a)} \int_0^\infty \frac{t^{a-1}}{\{x+t\}^{a-b+1}} \exp[-t] dt .$$

Set $x = sA$, where $s > 0$ is real. Thus,

$$(25) \quad U(a ; b ; sA) = \frac{s^{1-b} A^{1-b}}{\Gamma(a)} \int_0^\infty \frac{t^{a-1}}{\{sA+t\}^{a-b+1}} \exp[-t] dt .$$

Set $t = sv$, $s > 0$ as above.

Thus,

$$(26) \quad U(a ; b ; sA) = \frac{A^{1-b}}{\Gamma(a)} \int_0^{\infty} \frac{v^{a-1}}{\{A+v\}^{a-b+1}} \exp[-sv] dv .$$

For our purposes,

(i) $b = a$, and

(ii) A is a real valued function of arbitrary parameters,
such that $A > 0$.

Thus, for $a > 0$, $A > 0$, we have

$$(27) \quad U(a ; a ; sA) = \frac{A^{1-a}}{\Gamma(a)} \int_0^{\infty} \frac{v^{a-1}}{\{A + v\}} \exp[-sv] dv ,$$

or

$$(28) \quad \int_0^{\infty} \frac{v^{a-1}}{\{A+v\}} \exp[-sv] dv = \Gamma(a) A^{a-1} U(a ; a ; sA) .$$

Now consider the behaviour of $U(a ; a ; sA)$ as $s \rightarrow \infty$. It can be shown that

$$(29) \quad U(a ; a ; sA) = \frac{1}{s^a A^a} \{1 + O(\frac{1}{sA})\}$$

and thus,

$$(30) \quad \int_0^{\infty} \frac{v^{a-1}}{\{A+v\}} \exp[-sv] dv = \frac{\Gamma(a)}{A s^a} \{1 + O(\frac{1}{sA})\} .$$

Application of Theorem I to $U\left(\lambda+q ; \lambda+q ; \frac{s(\alpha-r)^2}{1-r^2}\right), q = 0, 1, \dots$

We consider the integral

$$(31) \quad \int_0^\infty \frac{v^{\lambda+q-1}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv] dv = \Gamma(\lambda+q) \left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{\lambda+q-1} U\left(\lambda+q, \lambda+q ; \frac{s(\alpha-r)^2}{1-r^2}\right).$$

For any fixed $v_0 \neq 0$ on D lying between 0 and ∞ , $\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}$ is uniformly bounded away from zero for all $v \in [v_0, \infty)$ i.e., uniform in the parameter r . [We recall that r varies continuously over the set $\{\xi : |\xi| < 1 \text{ and } \xi \neq \alpha\}$, where α is fixed.] Thus $\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}^{-1}$ is uniformly bounded for all v in $[v_0, \infty)$. Thus the integral

$$(32) \quad \int_0^\infty \frac{v^{\lambda+q-1}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv] dv$$

with the identification

$$f^*(v) = \left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}^{-1}$$

and

$$\psi(v) \equiv 1$$

satisfies all the conditions of Theorem I.

Thus,

$$(33) \quad \int_{\substack{v_0 \\ [D]}}^{\infty} \frac{v^{\lambda+q-1}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv] dv \approx 0$$

and

$$(34) \quad \int_{\substack{0 \\ [D]}}^{\infty} \frac{v^{\lambda+q-1}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv] dv \approx \int_{\substack{0 \\ [D]}}^{v_0} \frac{v^{\lambda+q-1}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv] dv,$$

uniformly in s and r , by virtue of the uniform boundedness of

$$\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}^{-1} \text{ for all } v \text{ in } [v_0, \infty).$$

Application of Theorem I to the integral I^* .

We consider the integral

$$(35) \quad I^* = \int_{\substack{0 \\ [D]}}^1 \frac{v^{\lambda-1} f(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv \psi(v)] dv \text{ of equation (18)}$$

which satisfies conditions (17), (i) - (iv).

With the identification

$$f^*(v) = \frac{f(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}},$$

I^* satisfies the conditions of Theorem I.

Thus,

$$(36) \quad \int_{\substack{v_0 \\ [D]}}^1 \frac{v^{\lambda-1} f(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv \psi(v)] dv \approx 0 ,$$

and

$$(37) \quad I^* = \int_{\substack{0 \\ [D]}}^1 \frac{v^{\lambda-1} f(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv \psi(v)] dv$$

$$\approx \int_{\substack{0 \\ [D]}}^{v_0} \frac{v^{\lambda-1} f(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv \psi(v)] dv ,$$

uniformly in s and r by virtue of the uniform boundedness of

$$\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}^{-1} \text{ for all } v \text{ in } [v_0, 1].$$

The Asymptotic Expansion for the Integral I^* .

We consider the integral

$$(38) \quad \int_{\substack{0 \\ [D]}}^{v_0} \frac{v^{\lambda-1} f(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv \psi(v)] dv ,$$

where we choose v_0 fixed such that $f(v)$ and $\psi(v)$ are regular for

$|v| \leq 2|v_0| < 1$. Write

$$(39) \quad \exp[-sv \psi(v)] = \exp[-sv] \exp[sv \{1-\psi(v)\}] .$$

Then,

$$\begin{aligned} (40) \quad \int_0^{v_0} \frac{v^{\lambda-1} f(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv \psi(v)] dv \\ = \int_0^{v_0} \frac{v^{\lambda-1} f(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[sv \{1-\psi(v)\}] \exp[-sv] dv \\ = \int_0^{v_0} \frac{v^{\lambda-1} f(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} G(sv, v) \exp[-sv] dv , \end{aligned}$$

where,

$$(41) \quad G(sv, v) = f(v) \exp[sv \{1 - \psi(v)\}] .$$

Now put $sv = w$ and treat $G(w, v)$ as a function of two variables. Thus for $|v| \leq 2|v_0|$

$$(42) \quad f(v) = \sum_{n=0}^{\infty} b_n v^n$$

and,

$$(43) \quad 1-\psi(v) = 1 + \frac{\log(1-v)}{v} = - \sum_{n=1}^{\infty} \frac{v^n}{n+1} ,$$

and,

$$\begin{aligned}
 (44) \quad \exp[w \{1-\psi(v)\}] &= \sum_{n=0}^{\infty} \frac{w^n \{1-\psi(v)\}^n}{n!} \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n w^n}{n!} \left\{ \sum_{k=1}^{\infty} \frac{v^k}{k+1} \right\}^n,
 \end{aligned}$$

which, on rearrangement, gives the power series

$$(45) \quad \sum_{n=0}^{\infty} Q_n(w) v^n,$$

where $Q_n(w)$ is a polynomial in w of degree $\leq n$. Thus,

$$\begin{aligned}
 (46) \quad f(v) \exp[w\{1-\psi(v)\}] &= \left\{ \sum_{n=0}^{\infty} b_n v^n \right\} \left\{ \sum_{n=0}^{\infty} Q_n(w) v^n \right\} \\
 &= \sum_{n=0}^{\infty} P_n(w) v^n
 \end{aligned}$$

where,

$$\begin{aligned}
 (47) \quad P_n(w) &= \sum_{k=0}^n b_{n-k} Q_k(w) \\
 &= \sum_{k=0}^n c_k^{(n)} w^k.
 \end{aligned}$$

Since for $|v| \leq 2|v_0|$, all component series converge, ~~there~~ $\sum_{n=0}^{\infty} P_n(w) v^n$

converges to $f(v) \exp[w\{1-\psi(v)\}]$ and is the Taylor series expansion of $G(w,v)$.

Thus the finite Taylor expansion is

$$(48) \quad G(w,v) = \sum_{n=0}^N P_n(w) v^n + V_N \quad |v| \leq |v_0|$$

where,

$$V_N = f_N(w,v) v^{N+1} ,$$

and,

$$f_N(w,v) = \sum_{n=N+1}^{\infty} P_n(w) v^{n-N-1} .$$

Now, $G(w,v)$ is uniformly bounded and regular in v on D for $|v| \leq 2|v_0|$ and all s . Thus f_N is a uniformly bounded and regular function of v for $|v| \leq 2|v_0|$ on D .

By means of the Cauchy inequalities the magnitude of the remainder V_N may also be expressed by means of an order relation.

Since $\psi(v)$ is regular for $|v| \leq |v_0|$ and $\psi(0) = 1$
 $|1-\psi(v)| \leq K|v_0|$, for some fixed $K > 0$, uniformly in v for $|v| \leq 2|v_0|$,
providing $|v_0|$ is taken to be fixed and sufficiently small. Thus

$$(49) \quad v_N = O(v^{N+1} \exp[K|v_0| |w|]) \quad \text{as } v \rightarrow 0 \quad (*)$$

Thus, returning to (40) we have

$$(50) \quad \int_0^{v_0} \frac{v^{\lambda-1}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} G(sv, v) \exp[-sv] dv$$

$$= \sum_{n=0}^N \int_0^{v_0} \frac{P_n(sv) v^{\lambda+n-1}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv] dv$$

$$+ \int_0^{v_0} \frac{v^{\lambda-1}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} v_N \exp[-sv] dv .$$

(*)

The Bachman-Landau $O - o$ notation:

Let x be a variable element of a topological T_2 -space, and let it range over a set R , let x_0 be a cluster point of R (which may or may not belong to R). $\varphi(x)$ and $\psi(x)$ denote real- or complex-valued functions defined for x in R . We write

(i) $\varphi = O(\psi)$ in R , if there exists a constant (i.e. a number independent of x) A such that

$$|\varphi| \leq A|\psi| \quad \text{for all } x \text{ in } R .$$

(ii) $\varphi = O(\psi)$ as $x \rightarrow x_0$, if there exists a constant A and a neighbourhood $N_A(x_0)$ of x_0 such that

$$|\varphi| \leq A|\psi| \quad \text{for all } x \text{ in } R \cap N_A(x_0) .$$

(iii) $\varphi = o(\psi)$ as $x \rightarrow x_0$, if for any given $\epsilon > 0$ there exists a neighbourhood $N_\epsilon(x_0)$ of x_0 such that

$$|\varphi| \leq \epsilon|\psi| \quad \text{for all } x \text{ in } R \cap N_\epsilon(x_0) .$$

Consider first

$$\begin{aligned}
 (51) \quad & \sum_{n=0}^N \int_{[D]}^{v_0} \frac{P_n(sv) v^{\lambda+n-1}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv] dv \\
 &= \sum_{n=0}^N \sum_{k=0}^n c_k^{(n)} s^k \int_{[D]}^{v_0} \frac{v^{\lambda+n+k-1}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv] dv .
 \end{aligned}$$

By equation (34), we have

$$\begin{aligned}
 (52) \quad & s^k \int_{[D]}^{v_0} \frac{v^{\lambda+n+k-1}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv] dv \\
 & \approx s^k \int_{[D]}^{\infty} \frac{v^{\lambda+n+k-1}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv] dv
 \end{aligned}$$

and by equation (28)

$$\begin{aligned}
 (53) \quad & \int_{[D]}^{\infty} \frac{v^{\lambda+n+k-1}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv] dv \\
 &= \Gamma(\lambda+n+k) \left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{\lambda+n+k-1} U(\lambda+n+k; \lambda+n+k; \frac{s(\alpha-r)^2}{1-r^2}) .
 \end{aligned}$$

Thus,

$$\begin{aligned}
 (54) \quad & \sum_{n=0}^N \sum_{k=0}^n c_k^{(n)} s^k \int_0^{v_0} \frac{v^{\lambda+n+k-1}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv] dv \\
 & \approx \sum_{n=0}^N \sum_{k=0}^n c_k^{(n)} \Gamma(\lambda+n+k) \left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{\lambda+n+k-1} \\
 & \quad s^k U(\lambda+n+k ; \lambda+n+k ; \frac{s(\alpha-r)^2}{1-r^2}) .
 \end{aligned}$$

Now the order of magnitude of

$$\sum_{k=0}^n c_k^{(n)} (\lambda+n+k) \left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{\lambda+n+k-1} s^k U(\lambda+n+k ; \lambda+n+k ; \frac{s(\alpha-r)^2}{1-r^2})$$

is uniformly (in s only)

$$O \left(\frac{1}{s^{\lambda+n}} \right) , \text{ as } s \rightarrow \infty ,$$

since (c.f. equation (29)) ,

$$\begin{aligned}
 (55) \quad & U(\lambda+n+k ; \lambda+n+k ; \frac{s(\alpha-r)^2}{1-r^2}) \\
 & = \left\{ \frac{1-r^2}{(\alpha-r)^2} \right\}^{\lambda+n+k} \frac{1}{s^{\lambda+n+k}} \left\{ 1 + O \left(\frac{1}{s \frac{(\alpha-r)^2}{1-r^2}} \right) \right\} .
 \end{aligned}$$

Now consider the remainder term

$$\begin{aligned}
 (56) \quad R_N &= \int_{[D]}^{v_0} \frac{v^{\lambda-1}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} v_N \exp[-sv] dv \\
 &= \int_{[D]}^{v_0} \frac{v^{\lambda+N}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} f_N(sv, v) \exp[-sv] dv ,
 \end{aligned}$$

where $|f_N| \leq M$, for some fixed $M > 0$ uniformly for $|v| \leq 2|v_0|$.

Thus,

$$\begin{aligned}
 (57) \quad & \left| \int_{[D]}^{v_0} \frac{v^{\lambda+N}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} f_N(sv, v) \exp[-sv] dv \right| \\
 & \leq M \left| \int_{[D]}^{v_0} \frac{v^{\lambda+N}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv] dv \right| \\
 & \leq M \left| \Gamma(\lambda+N+1) \left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{\lambda+N} U(\lambda+N+1; \lambda+N+1; \frac{s(\alpha-r)^2}{1-r^2}) \right| .
 \end{aligned}$$

Now by (29)

$$(58) \quad \left| U(\lambda+N+1; \lambda+N+1; \frac{s(\alpha-r)^2}{1-r^2}) \right| \leq m \left| \left\{ \frac{(1-r^2)}{(\alpha-r)^2} \right\}^{\lambda+N+1} \frac{1}{s^{\lambda+N+1}} \right|$$

for some fixed $m > 0$.

Thus,

$$(59) \quad \left| \int_0^{v_0} \frac{v^{\lambda+N}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} f_N(sv, v) \exp[-sv] dv \right|$$

$$\leq mM \left| \Gamma(\lambda+N+1) \frac{(1-r^2)}{(\alpha-r)^2} \frac{1}{s^{\lambda+N+1}} \right|.$$

Now consider

$$(60) \quad mM \left| \Gamma(\lambda+N+1) \frac{1-r^2}{(\alpha-r)^2} \frac{1}{s^{\lambda+N+1}} \right| \bigg/ \left| \frac{1}{s^{\lambda+N}} \right|$$

$$= mM \left| \Gamma(\lambda+N+1) \frac{1-r^2}{(\alpha-r)^2} \frac{1}{s} \right|$$

which $\rightarrow 0$ as $s \rightarrow \infty$.

Thus

$$R_N = o\left(\frac{1}{s^{\lambda+N}}\right)$$

where the order relation is uniform in s but not in r .

Thus,

$$\begin{aligned}
 (61) \quad & \int_0^{v_0} \frac{v^{\lambda-1}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} G(sv, v) \exp[-sv] dv \\
 & - \sum_{n=0}^N \sum_{k=0}^n c_k^{(n)} \Gamma(\lambda+n+k) \left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{\lambda+n+k-1} \\
 & \times s^k U\left(\lambda+n+k ; \lambda+n+k ; \frac{s(\alpha-r)^2}{1-r^2}\right) = o\left(\frac{1}{s^{\lambda+N}}\right),
 \end{aligned}$$

which, together with equation (37), implies

$$\begin{aligned}
 (62) \quad & I^* = \int_0^1 \frac{v^{\lambda-1} f(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv \psi(v)] dv \\
 & - \sum_{n=0}^N \sum_{k=0}^n c_k^{(n)} \Gamma(\lambda+n+k) \left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{\lambda+n+k-1} \\
 & \times s^k U\left(\lambda+n+k ; \lambda+n+k ; \frac{s(\alpha-r)^2}{1-r^2}\right) = o\left(\frac{1}{s^{\lambda+N}}\right),
 \end{aligned}$$

which implies that

$$\begin{aligned}
 (63) \quad & \int_0^1 \frac{v^{\lambda-1} f(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv \psi(v)] dv \\
 & \sim \sum_{n=0}^{\infty} \sum_{k=0}^n c_k^{(n)} \Gamma(\lambda+n+k) \left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{\lambda+n+k-1}
 \end{aligned}$$

$$s^k U\left(\lambda+n+k ; \lambda+n+k ; \frac{s(\alpha-r)^2}{1-r^2}\right)$$

with respect to the sequence $\{s^{-\lambda-n}\}$.

Thus from (62) we have

$$\begin{aligned}
 & \int_0^1 \frac{v^{\lambda-1} f(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv \psi(v)] dv \\
 & = \sum_{n=0}^N \sum_{k=0}^n c_k^{(n)} \Gamma(\lambda+n+k) \left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{\lambda+n+k-1} \\
 & \times s^k U\left(\lambda+n+k ; \lambda+n+k ; \frac{s(\alpha-r)^2}{1-r^2}\right) = o\left(\frac{1}{s^{\lambda+N}}\right) .
 \end{aligned}$$

Thus,

$$\begin{aligned}
 (64) \quad & \int_0^1 \frac{v^{\lambda-1} f(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv \psi(v)] dv \\
 & - \sum_{n=0}^{N-1} \sum_{k=0}^n c_k^{(n)} \Gamma(\lambda+n+k) \left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{\lambda+n+k-1} \\
 & \times s^k U\left(\lambda+n+k; \lambda+n+k; \frac{s(\alpha-r)^2}{1-r^2}\right) \\
 & = \sum_{k=0}^N c_k^{(N)} \Gamma(\lambda+N+k) \left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{\lambda+N+k-1} s^k U\left(\lambda+N+k; \lambda+N+k; \frac{s(\alpha-r)^2}{1-r^2}\right) \\
 & \quad + o\left(\frac{1}{s^{\lambda+N}}\right) .
 \end{aligned}$$

Now by (58),

$$\begin{aligned}
 (65) \quad & \sum_{k=0}^N c_k^{(N)} \Gamma(\lambda+N+k) \left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{\lambda+N+k-1} s^k U\left(\lambda+N+k; \lambda+N+k; \frac{s(\alpha-r)^2}{1-r^2}\right) \\
 & = o\left(\frac{1}{s^{\lambda+N}}\right) .
 \end{aligned}$$

Thus,

$$\begin{aligned}
 (66) \quad & \int_0^1 \frac{v^{\lambda-1} f(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv \psi(v)] dv \\
 & - \sum_{n=0}^N \sum_{k=0}^n c_k^{(n)} \Gamma(\lambda+n+k) \left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{\lambda+n+k-1} \\
 & s^k U\left(\lambda+n+k; \lambda+n+k; \frac{s(\alpha-r)^2}{1-r^2}\right) = O\left(\frac{1}{s^{\lambda+N+1}}\right)
 \end{aligned}$$

where the order relation is uniform in s but not in r .

A more exact result for the estimate of the order of magnitude of the error involved in stopping at $n = N$ in (63) may be obtained by use of equation (49). Thus

$$\begin{aligned}
 (67) \quad & \int_0^{v_0} \frac{v^{\lambda-1}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} V_N \exp[-sv] dv \\
 & = \int_0^{v_0} \frac{v^{\lambda-1}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \left[O(v^{N+1} \exp[K|v_0| |sv|]) \right] \exp[-sv] dv \\
 & = O\left(\int_0^{v_0} \frac{v^{\lambda+N}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv(1-Kv_0)] dv \right)
 \end{aligned}$$

where $v_0 > 0$ and $sv > 0$ for all v on D in $[0, v_0]$.

Now for a sufficiently small choice of v_0 , $1 - Kv_0 \geq \delta > 0$, for some fixed δ . Thus for some fixed $\epsilon > 0$, for $0 < v_0 \leq \epsilon$, we have

$$\begin{aligned}
 (68) \quad R_N &= O \left(\int_0^\epsilon \frac{v^{\lambda+N}}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv\delta] dv \right) \\
 &= O \left(\int_0^\infty \frac{w^{\lambda+N}}{\left\{ \frac{s\delta(\alpha-r)^2}{1-r^2} + w \right\}} \exp[-w] dw \right) / \delta^{\lambda+N} s^{\lambda+N}
 \end{aligned}$$

and since δ , λ and N are fixed,

$$\begin{aligned}
 (69) \quad &\int_0^\infty \frac{w^{\lambda+N}}{\left\{ \frac{s\delta(\alpha-r)^2}{1-r^2} + w \right\}} \exp[-w] dw / \delta^{\lambda+N} s^{\lambda+N} \\
 &= O \left(\int_0^\infty \frac{w^{\lambda+N}}{\left\{ \frac{s(\alpha-r)^2}{1-r^2} + w \right\}} \exp[-w] dw \right) / s^{\lambda+N}
 \end{aligned}$$

as $s \rightarrow \infty$.

Thus,

$$\begin{aligned}
 (70) \quad R_N &= O \left(\int_0^\infty \frac{w^{\lambda+N}}{\left\{ \frac{s(\alpha-r)^2}{1-r^2} + w \right\}} \exp[-w] dw \right) / s^{\lambda+N} \\
 &= O \left(\left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{\lambda+N} \Gamma(\lambda+N+1) U(\lambda+N+1; \lambda+N+1; \frac{s(\alpha-r)^2}{1-r^2}) \right)
 \end{aligned}$$

(by equation (24)).

Thus we have shown

$$\begin{aligned}
 (71) \quad & \int_0^1 \frac{v^{\lambda-1} f(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv \psi(v)] dv \\
 & [D] \\
 & = \sum_{n=0}^N \sum_{k=0}^n c_k^{(n)} \Gamma(\lambda+n+k) \left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{\lambda+n+k-1} \\
 & \quad \times s^k U\left(\lambda+n+k; \lambda+n+k; \frac{s(\alpha-r)^2}{1-r^2}\right) \\
 & \quad + O\left(\left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{\lambda+N} \Gamma(\lambda+N+1) U\left(\lambda+N+1; \lambda+N+1; \frac{s(\alpha-r)^2}{1-r^2}\right)\right).
 \end{aligned}$$

Now by equations (2), (12) and (17) we have

$$\begin{aligned}
 (72) \quad I(r) &= \int_{-1}^{+1} \frac{\chi(t)}{\alpha-r-i\sqrt{1-r^2}t} (1-t^2)^s dt \\
 & [L^*] \\
 &= \frac{\alpha-r}{1-r^2} \int_0^1 \frac{v^{-\frac{1}{2}} f_1(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} (1-v)^s dv \\
 & \quad - \int_0^1 \frac{v^{\frac{1}{2}} f_2(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} (1-v)^s dv \\
 & [D]
 \end{aligned}$$

where,

$$(i) \quad v = t^2$$

$$(ii) \quad f_1(v) = \frac{1}{2}[\chi(t) + \chi(-t)]$$

$$(iii) \quad f_2(v) = \frac{1}{2i\sqrt{1-r^2}} [\chi(t) - \chi(-t)]$$

and,

$$(iv) \quad D \text{ is the } v\text{-image of } L^* .$$

Further,

$$(73) \quad \int_{[D]}^1 \frac{v^{-\frac{1}{2}} f_1(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} (1-v)^s dv = \int_{[D]}^1 \frac{v^{-\frac{1}{2}} f_1(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv \psi(v)] dv$$

and

$$(74) \quad \int_{[D]}^1 \frac{v^{\frac{1}{2}} f_2(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} (1-v)^s dv = \int_{[D]}^1 \frac{v^{\frac{1}{2}} f_2(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} \exp[-sv \psi(v)] dv.$$

Thus, by equations (71) and (73)

$$(75) \quad \int_{[D]}^1 \frac{v^{-\frac{1}{2}} f_1(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} (1-v)^s dv = \sum_{n=0}^N \sum_{k=0}^n c_k^{(n,1)} \Gamma(n+k+\frac{1}{2}) \left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{n+k-\frac{1}{2}} \\ \times s^k U\left(n+k+\frac{1}{2}; n+k+\frac{1}{2}; \frac{s(\alpha-r)^2}{1-r^2}\right) \\ + O\left(\left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{N+\frac{1}{2}} \Gamma(N+\frac{3}{2}) U\left(N+\frac{3}{2}; N+\frac{3}{2}; \frac{s(\alpha-r)^2}{1-r^2}\right)\right),$$

and, by equations (71) and (74)

$$\begin{aligned}
 (76) \quad & \int_0^1 \frac{v^{\frac{1}{2}} f_2(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} (1-v)^s dv \\
 & = \sum_{n=0}^{N-1} \sum_{k=0}^n c_k^{(n,2)} \Gamma(n+k+\frac{3}{2}) \left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{n+k+\frac{1}{2}} \\
 & \quad s^k U \left(n+k+\frac{3}{2} ; n+k+\frac{3}{2} ; \frac{s(\alpha-r)^2}{1-r^2} \right) \\
 & \quad + O \left(\left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{N+\frac{1}{2}} \Gamma(N+\frac{3}{2}) U(N+\frac{3}{2} ; N+\frac{3}{2} ; \frac{s(\alpha-r)^2}{1-r^2}) \right),
 \end{aligned}$$

where $c_k^{(n,1)}$ and $c_k^{(n,2)}$ are given by

$$(77) \quad \sum_{n=0}^{\infty} \sum_{k=0}^n c_k^{(n,1)} s^k v^{n+k} = f_1(v) \exp[sv \{1 - \psi(v)\}] ,$$

and

$$(78) \quad \sum_{n=0}^{\infty} \sum_{k=0}^n c_k^{(n,2)} s^k v^{n+k} = f_2(v) \exp[sv \{1 - \psi(v)\}] .$$

Now changing the index of summation in (76) and (78) we have, corresponding to (76)

$$\begin{aligned}
 (79) \quad & \int_0^1 \frac{v^{\frac{1}{2}} f_2(v)}{\left\{ \frac{(\alpha-r)^2}{1-r^2} + v \right\}} (1-v)^s dv \\
 & [D] \\
 & = \sum_{n=1}^N \sum_{k=0}^{n-1} c_k^{(n-1,2)} \Gamma(n+k+\frac{1}{2}) \left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{n+k-\frac{1}{2}} s^k U \left(n+k+\frac{1}{2}; n+k+\frac{1}{2}; \frac{s(\alpha-r)^2}{1-r^2} \right) \\
 & + O \left(\left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{N+\frac{1}{2}} \Gamma(N+\frac{3}{2}) U(N+\frac{3}{2}; N+\frac{3}{2}; \frac{s(\alpha-r)^2}{1-r^2}) \right)
 \end{aligned}$$

where $c_k^{(n-1,2)}$ is given by (the correspondent to (78))

$$(80) \quad \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} c_k^{(n-1,2)} s^k v^{n+k-1} = f_2(v) \exp[sv \{1 - \psi(v)\}] .$$

Thus (72) becomes

$$\begin{aligned}
 (81) \quad & \int_{-1}^{+1} \frac{\chi(t)}{\alpha-r-i\sqrt{1-r^2} t} (1-t^2)^s dt \\
 & [L^*] \\
 & = \frac{\alpha-r}{1-r^2} \sum_{n=0}^N \sum_{k=0}^n c_k^{(n,1)} \Gamma(n+k+\frac{1}{2}) \left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{n+k-\frac{1}{2}} s^k U \left(n+k+\frac{1}{2}; n+k+\frac{1}{2}; \frac{s(\alpha-r)^2}{1-r^2} \right) \\
 & - \sum_{n=1}^N \sum_{k=0}^{n-1} c_k^{(n-1,2)} \Gamma(n+k+\frac{1}{2}) \left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{n+k-\frac{1}{2}} s^k U \left(n+k+\frac{1}{2}; n+k+\frac{1}{2}; \frac{s(\alpha-r)^2}{1-r^2} \right) \\
 & + O \left(\left\{ \frac{(\alpha-r)^2}{1-r^2} \right\}^{N+\frac{1}{2}} \Gamma(N+\frac{3}{2}) U(N+\frac{3}{2}; N+\frac{3}{2}; \frac{s(\alpha-r)^2}{1-r^2}) \right) .
 \end{aligned}$$

APPENDIX IV

GRAPHICAL RESULTS

In this appendix seven graphs are given. The purpose of drawing these graphs is to indicate the general trend and shape of the distributions obtained in Chapter three: specifically, the results given in equations (3.2.14) and (3.2.15) are for the circular case, and in equations (3.2.16) and (3.2.17) for the stationary case.

The seven graphs are divided into three groups. Each group is preceded by a set of prefatory remarks regarding the characteristic features brought into prominence by that particular group.

As regards the graphs themselves, the variable r varies over the closed interval $[-1,+1]$, and is represented along the horizontal axis. In Groups A and B, $h(r)$ and $H(r)$ [Daniels' density function and our distribution function respectively] are represented along the vertical axis. The scale for $h(r)$ being given on the left and for $H(r)$ being given on the right. In group C only the distribution function $H(r)$ is represented along the vertical axis.

GROUP A.

Each of the following three graphs contains a tracing of the function given by equations (3.2.14) and (3.2.15), which is the approximate Distribution function for the circular sample serial correlation coefficient. Included also is a tracing of the function

$$\frac{\Gamma(n+1)}{\sqrt{\pi} \Gamma(\frac{n}{2} + \frac{1}{2})} \quad \frac{(1-r^2)^{\frac{n}{2} - \frac{1}{2}}}{(1-2\alpha r + \alpha^2)^{\frac{n}{2}}}$$

which is the approximate Density function for the same sample serial correlation coefficient as derived in DANIELS (1956).

The three cases considered are

- (i) $n = 10, \quad \alpha = -0.5 .$
- (ii) $n = 20, \quad \alpha = 0.0 .$
- (iii) $n = 30, \quad \alpha = +0.5 .$

These three cases give a fair cross-section of the behavior of the derived approximate distribution function, and its relationship with Daniels' result.

The most prominent feature exhibited by the three graphs is that the skewness of the distribution in question is a function of α .

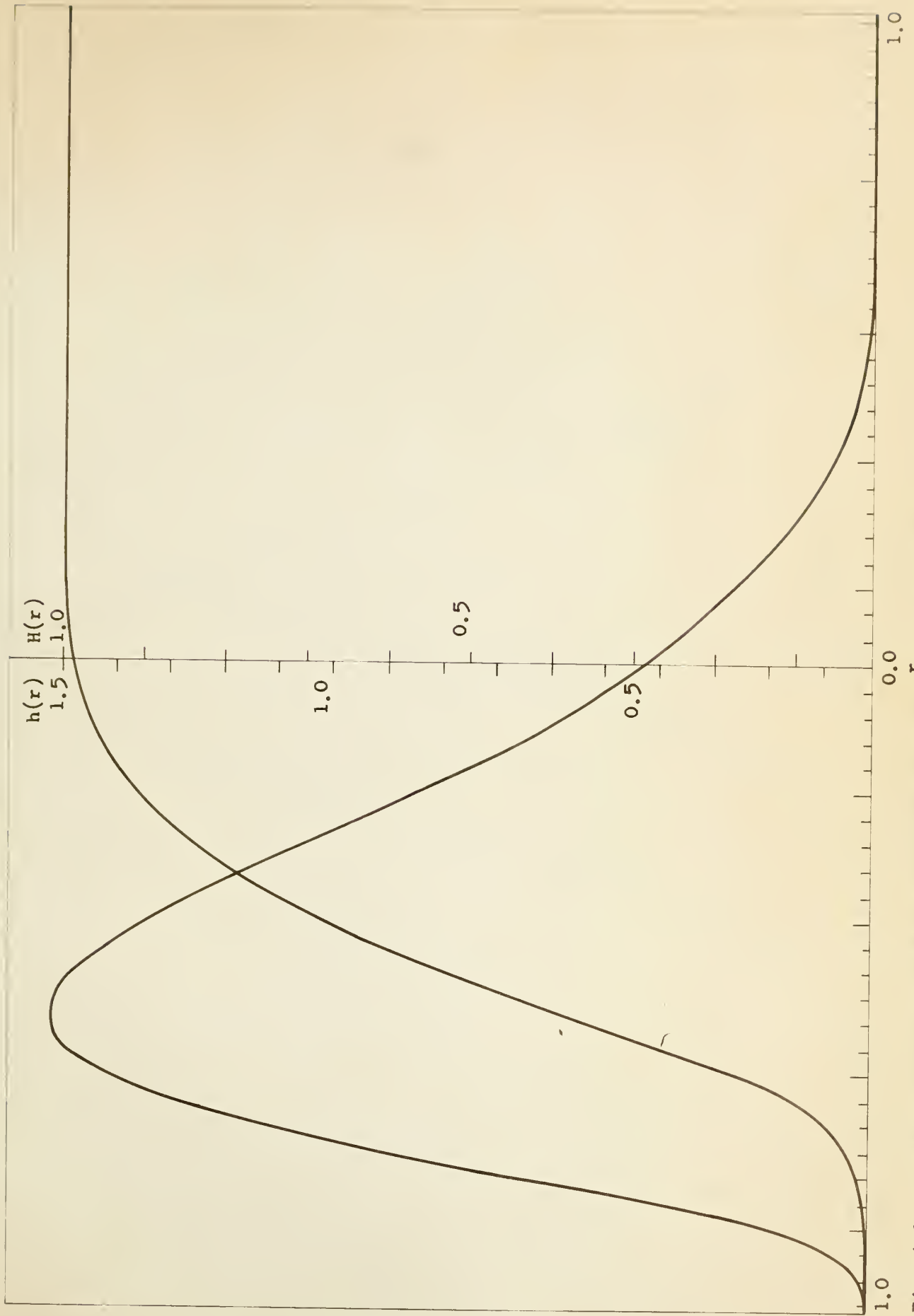


Fig. A.1 Distribution and Density Functions for $n = 10$, $\alpha = -0.5$

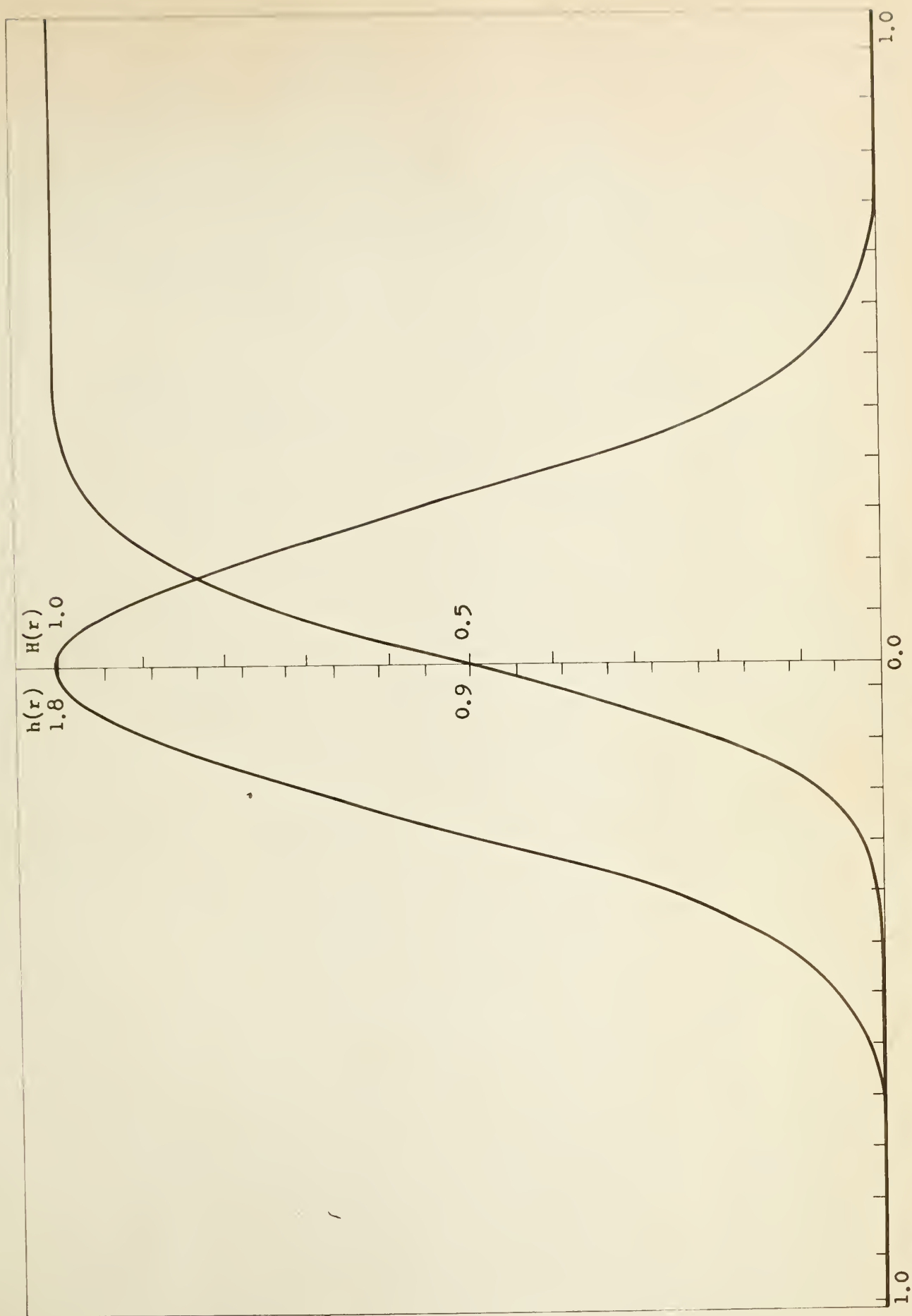


Fig. A.2 Distribution and Density Functions for $n = 20$, $\alpha = 0.0$

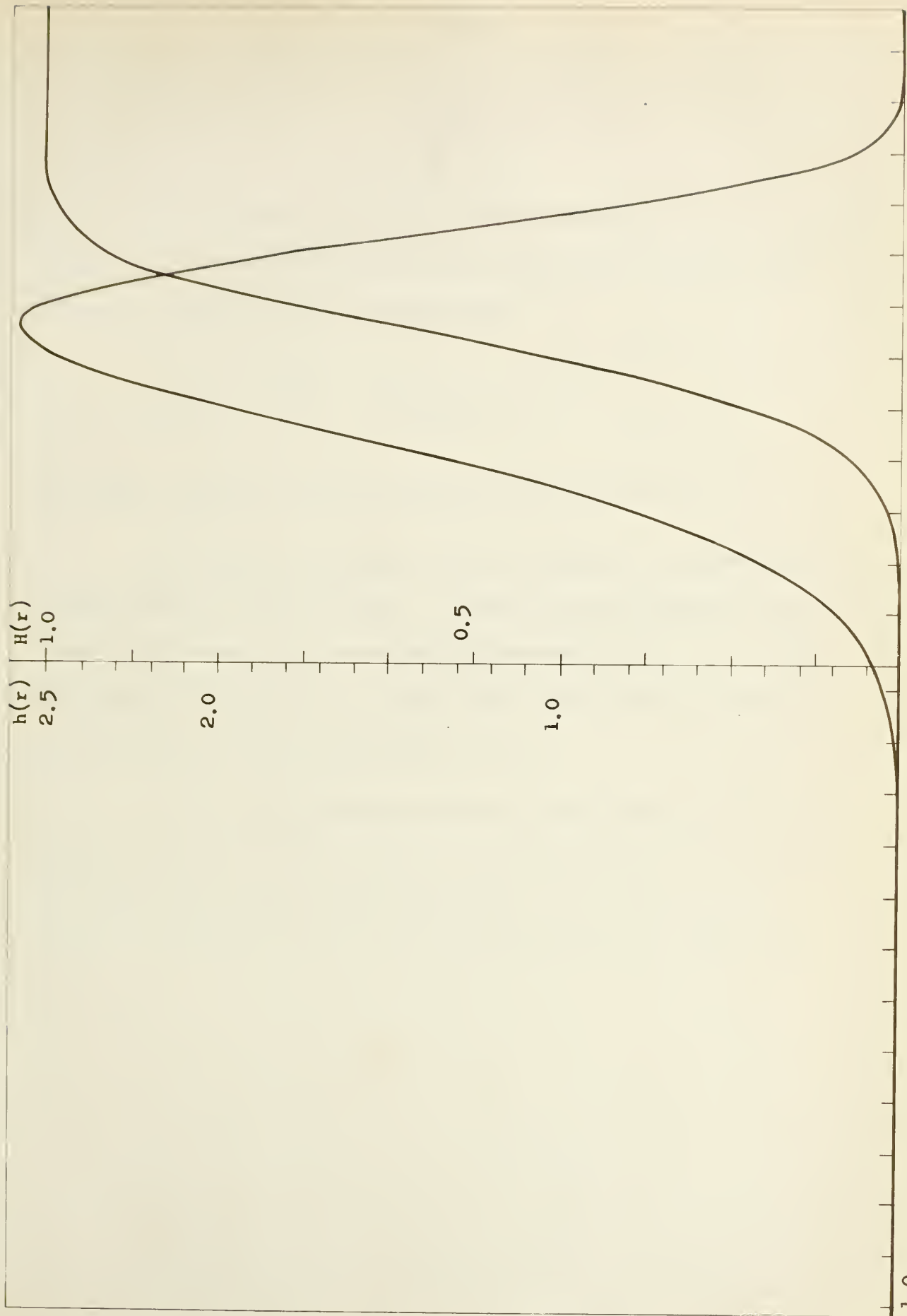


Fig. 4.3 Distribution and Density Functions for $n = 30$, $\alpha = +0.5$

GROUP B.

The following two graphs, also for the circular case, show the effect on the Kurtosis of distribution in question by changes in the size of n . The two cases considered are

- (i) $\alpha = 0.0$, $n = 10$
- (ii) $\alpha = 0.0$, $n = 20$.

The graphs indicate that the Kurtosis is a function of n .

Both group A and group B serve to indicate the relationship between Daniels' result and ours. Although the relationship is approximate, there seems to be an excessive discrepancy near $r = -1$. This discrepancy we attribute to the fact that neither Daniels' result nor ours is reliable in the regions near $r = \pm 1$.

A similar relationship between the Distribution function and Daniels' Density function exists for the stationary case. Because of this similarity between the stationary and circular cases we have omitted the sketches (as in groups A and B) for the stationary case.

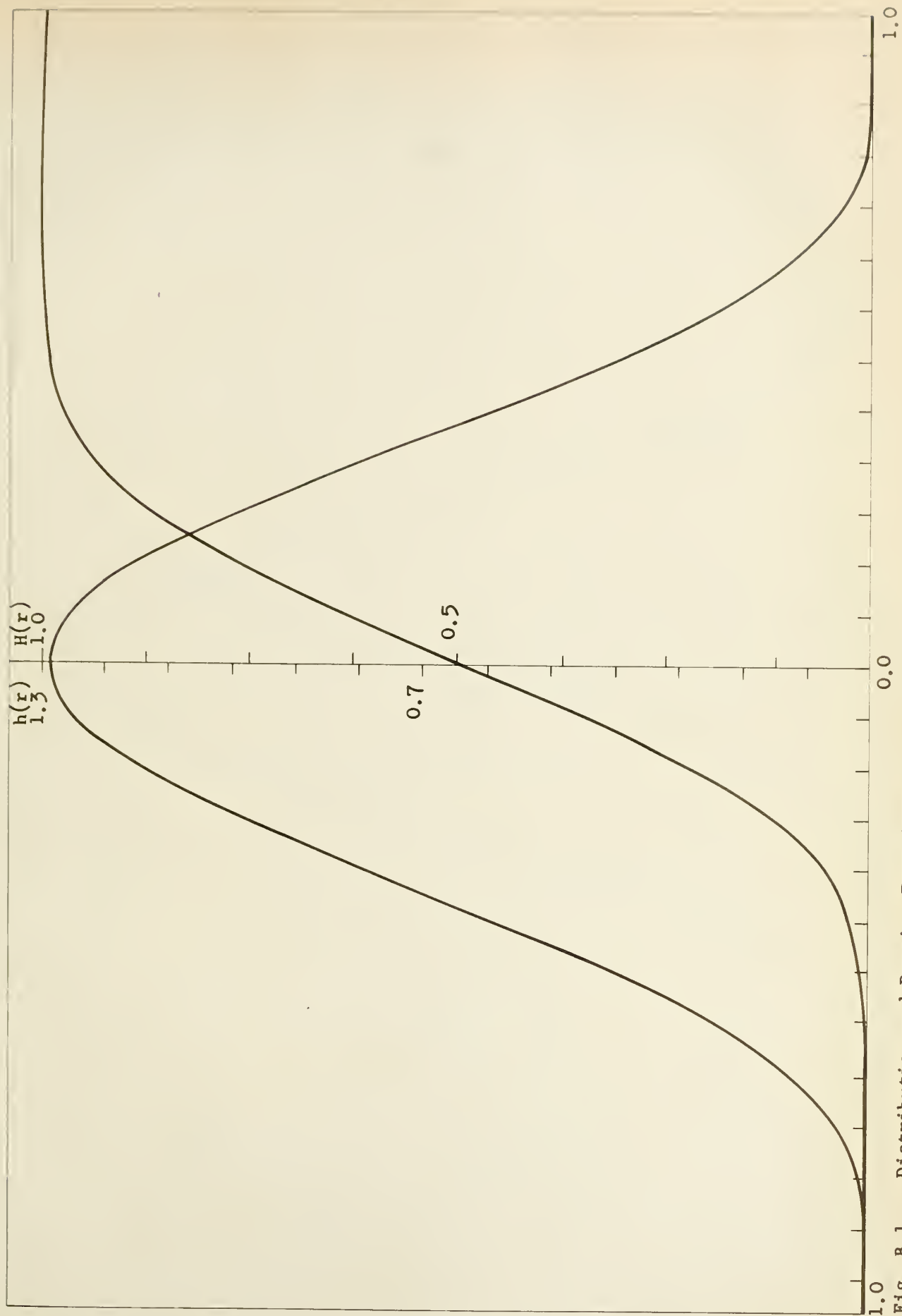


Fig. B.1 Distribution and Density Functions for $n = 10$, $\alpha = 0.0$

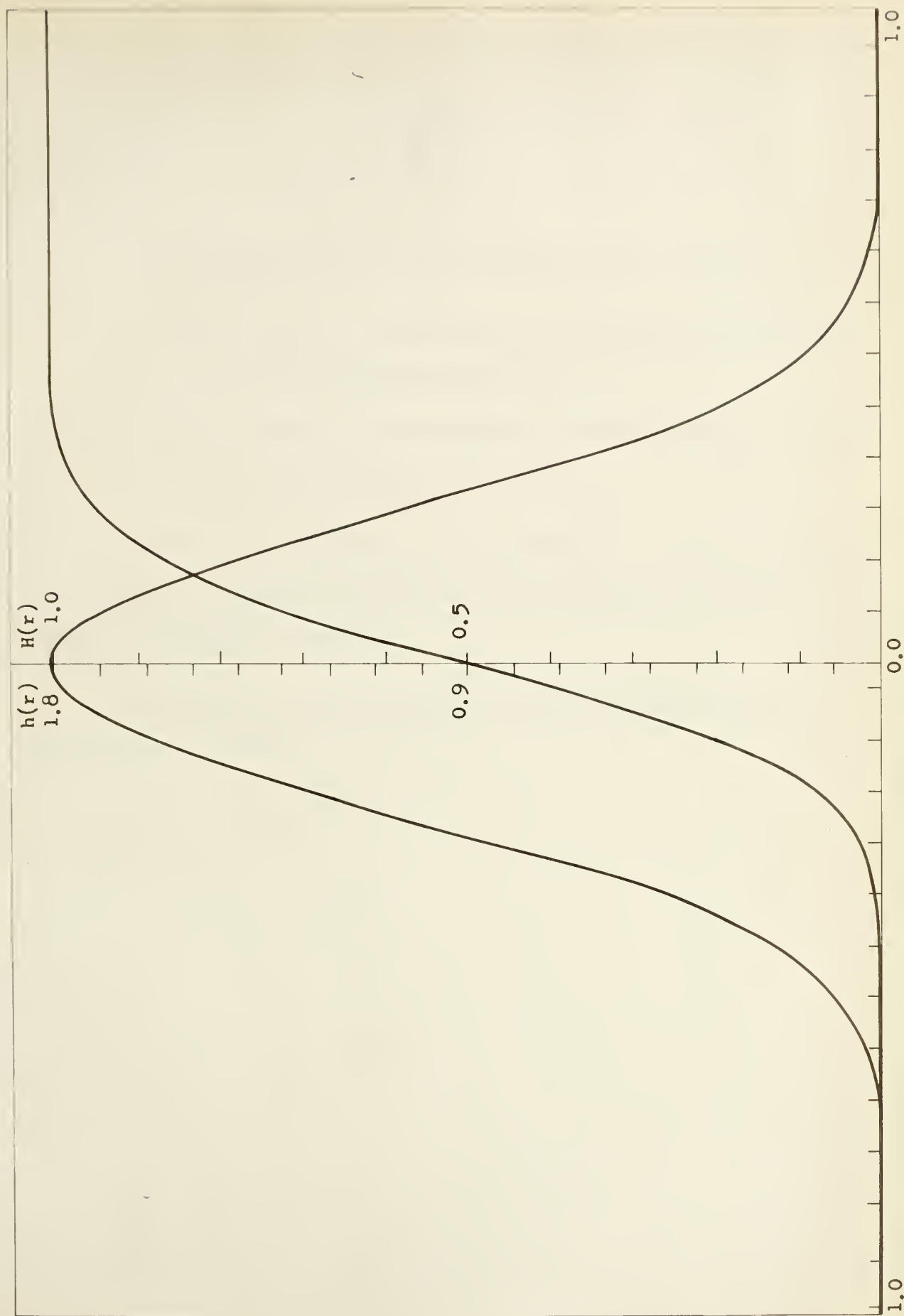


Fig. B.2 Distribution and Density Functions for $n = 20$, $\alpha = 0.0$

GROUP C.

For the stationary case, the following graphs indicate

- (i) for fixed n , the variation in the distribution function for three different values of α .
- (ii) for fixed α , the variation in the distribution function for three different values of n .

To bring into focus the obvious variations a system of superposition has been adopted, the various cases considered are clearly marked on the respective graphs.

The results of these two graphs overlap somewhat with those given by groups A and B.

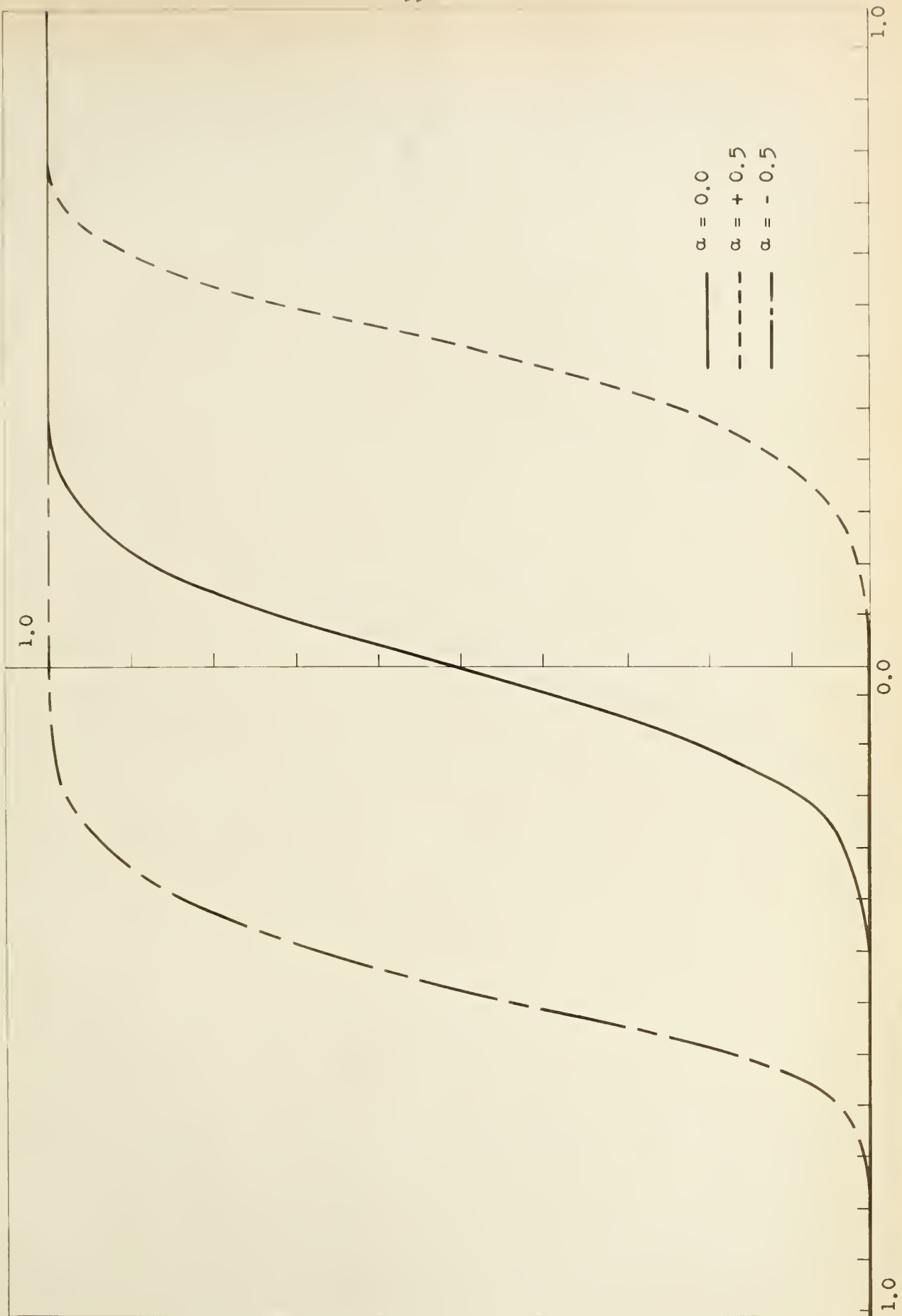


Fig. C.1 Distribution Functions for $n = 20$



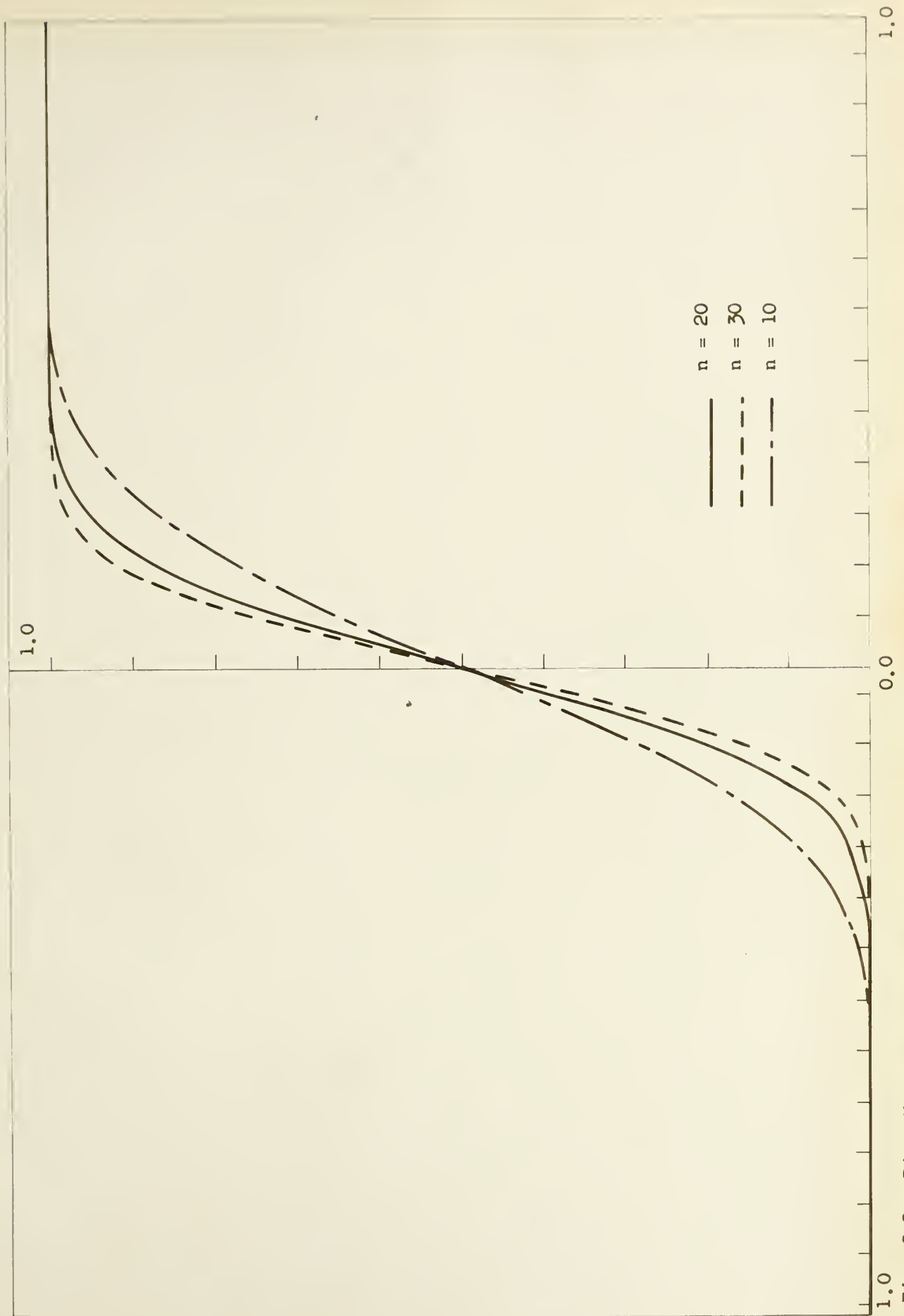


Fig. C.2 Distribution Functions for $\alpha = 0.0$

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